Computer Arithmetic
Ch 8

ALU

Integer Representation

Integer Arithmetic

Floating-Point Representation

Floating-Point Arithmetic

Arithmetic Logical Unit (ALU) (2)

(aritmeettis-looginen yksikkö)

- Does all "work" in CPU

Rest is management!

  - integer & floating point arithmetic's
  - copy values from one register to another
  - comparisons
  - left and right shifts
  - branch and jump address calculations
  - load/store address calculations
- Control signals from CPU control unit
  - what operation to perform and when

ALU Operations (5)

- Data from/to internal registers (latches)
  - input data may have been copied from normal registers, or it may have come from memory
  - output data may go to normal registers, or to memory

Fig. 8.1

- Wait for maximum gate delay
- Result is ready
- Result may (also) be in flags

(lipuke)

- Flags may cause an interrupt

Integer Representation (8)

Everything with 0 and 1
- no plus/minus signs
- no decimal periods
  - assumed "on the right"

+32 +16 +8 +1

57 = 0011 1001 = 0x39

hexadecimal presentation

3*16 +9*1

- Unsigned integers
- Positive numbers easy
  - normal binary form
- Negative numbers
  - sign-magnitude
  - two's complement

sign bit = MSB
= most significant bit

-57 = 1011 1001

-57 = 1100 0111

+1

"sign" bit

complements

Twos Complement

(kahden komplementti)

- Most used
- Have space for 8 bits?
  - use 7 bits for data and 1 bit for sign

+2 =	0000 0010
+1 =	0000 0001
0 =	0000 0000
-1 =	1111 1111
-2 =	1111 1110

  - just like in sign-magnitude or in one's complement (but presentation is different)

ones complement: -0 = 1111 11<u>11</u>

Why Two's Complement Presentation? (4)

- Math is easy to implement
  - subtraction becomes addition

$X\text{-}Y = X + (\text{-}Y)$

easy to do,
simple circuit

- Have just one zero
  - comparisons to zero easy
- Easy to expand to presentation with more bits
  - simple circuit

$57 = \underline{0}011\ 1001 = \underline{0000\ 0000}\ \underline{0}011\ 1001$

$-57 = \underline{1}100\ 0111 = \underline{1111\ 1111}\ \underline{1}100\ 0111$

sign extension

Why Two's Complement Presentation? (3)

- Range with n bits: $-2^{n-1} \ldots 2^{n-1} - 1$

  8 bits: $-2^{7} \ldots 2^{7} - 1 = -128 \ldots 127$
  32 bits: $-2^{31} \ldots 2^{31} - 1 = -2\,147\,483\,648 \ldots 2\,147\,483\,647$

- Overflow easy to recognise
  – add positive & negative - no overflows
  – add 2 positive/negative numbers
    - if sign bit of result is different? $\Rightarrow$ overflow!

$$
\begin{array}{rl}
57 = & 0011\ 1001 \\
+\ 80 = & 0101\ 0000 \\
\hline
137 = & \underline{1}000\ 1001
\end{array}
$$

Why Two’s Complement Presentation? (5)

- Addition easy if one or both operands negative
  - treat them all as unsigned integers

Same circuit works for both (except for overflow check)

13 = 1101	-3 = 1101
+1 = 0001	+1 = 0001
14 = 1110	-2 = 1110
Digits represent 4 bit unsigned numbers	Digits represent 4 bit two’s complement numbers

+3 = 0011

$$\begin{array}{r} 1100 \\ +1 \\ \hline 1101 \end{array}$$

Integer Arithmetic Operations

- Negation $X = -Y$
- Addition $X = Y+Z$
- Subtraction $X = Y-Z$
- Multiplication $X = Y*Z$
- Division $X = Y / Z$

Integer Negation (6)

57 = 0011 1001

1100 0110

+1

1100 0111

- Step 1: negate all bits
- Step 2: add 1

Step 3: special cases

0 = 0000 0000

1111 1111

+1

-0 = <u>1</u> 0000 0000

- ignore carry bit
  - negate 0?
- check that sign bit really changes
  - can not negate smallest negative
  - results in exception?

-128 = <u>1</u>000 0000

bitwise not: 0111 1111

add 1: <u>1</u>000 0000

Integer Addition and Subtraction (4)

- Normal binary addition
  - 32 bit full adder?
- Ignore carry & monitor sign bit for overflow
- In case of SUB, complement 2nd operand
- 2 circuits
  - addition
  - complement

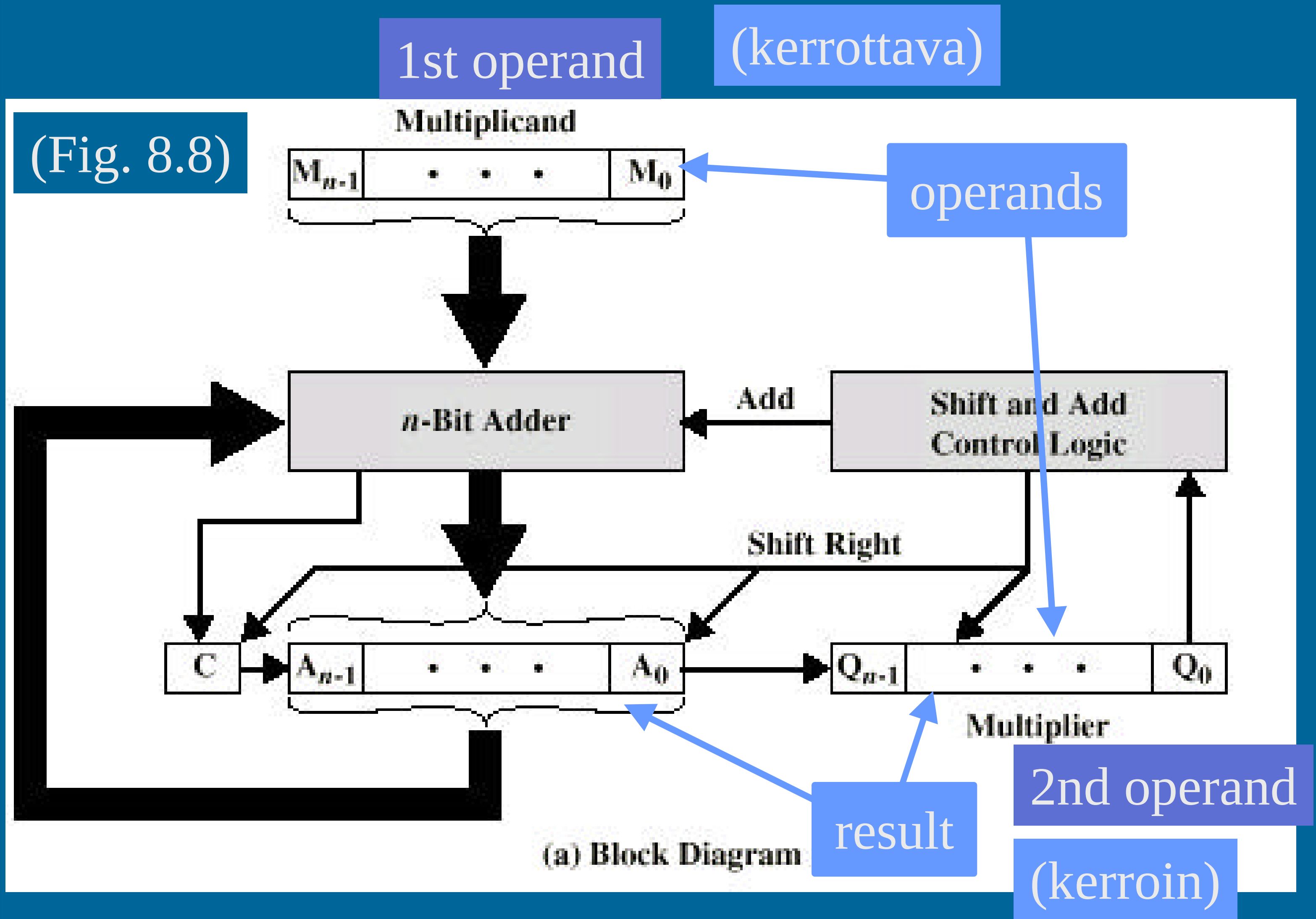
Fig. 8.6

Integer Multiplication (4)

- Complex
- Operands 32 bits ⇒ result 64 bits
- "Just like" you learned at school
  - optimised for binary data
    - it is easy to multiply with 0 or 1!
- Simpler case with unsigned numbers
  - simple circuits
    - adder
    - shifter
    - wires

Fig. 8.7

Unsigned Multiplication Example



Unsigned Multiplication Example (19)

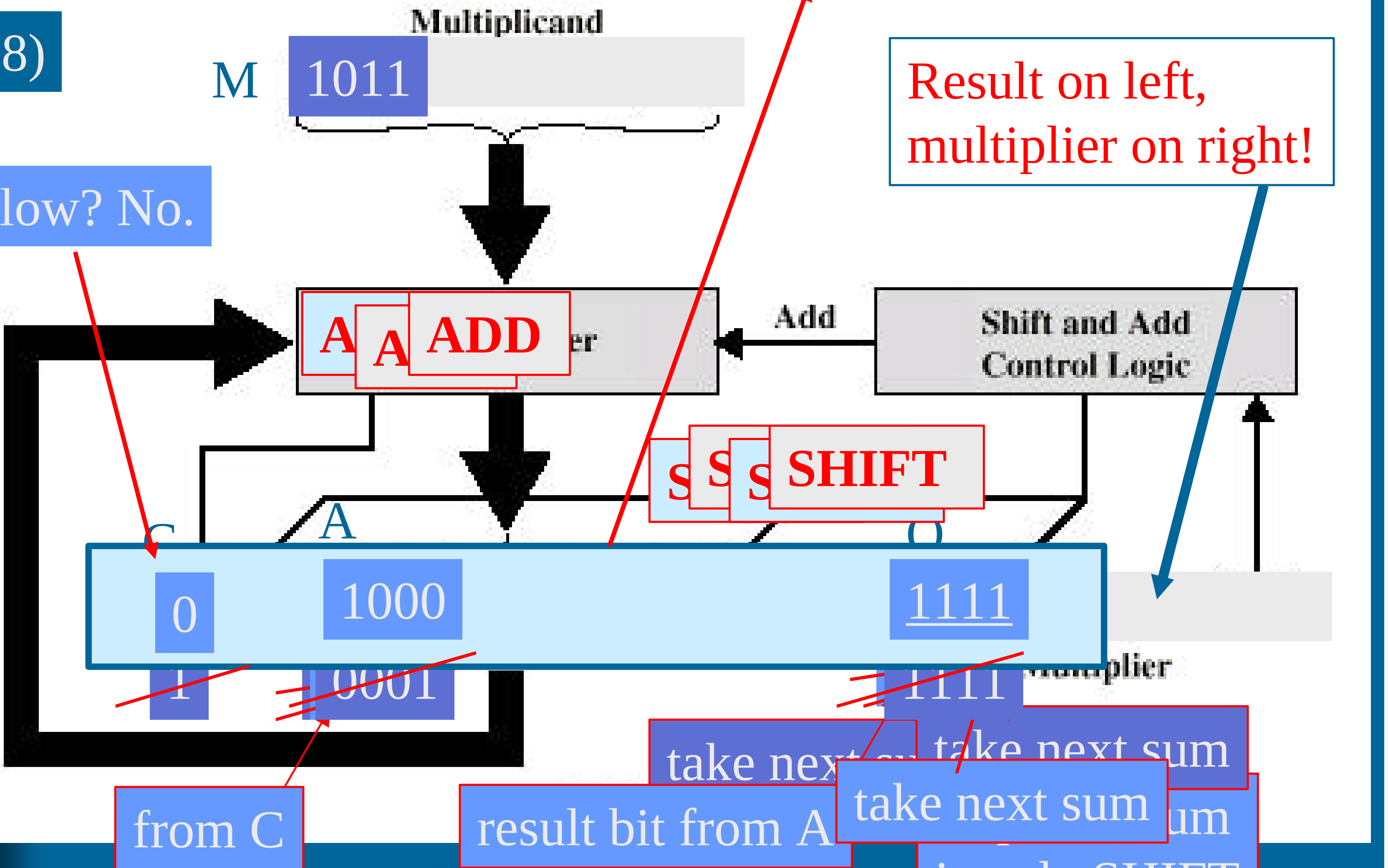
13 * 11 = ???

= 1000 1111 = 128+8+4+2+1 = 143

(Fig. 8.8)

Overflow? No.

Result on left, multiplier on right!



Multiplication with Negative Values

- Multiplication for unsigned numbers does not work for negative numbers
  - algorithm applies only for unsigned integer representation
  - not the same case as with addition
- Could do it all with unsigned values
  - change operands to positive values
  - do multiplication with positive values
  - negate result if needed
  - OK, but can do better, I.e., faster

The Gist in Booth's Algorithm (7)

Unsigned multiplication:

addition for every "1" bit
in multiplicand

5 * 7 ⇨ 0101 * 0111 ⇨

```
     0101
 +  01010
 + 010100
 = 100011
```

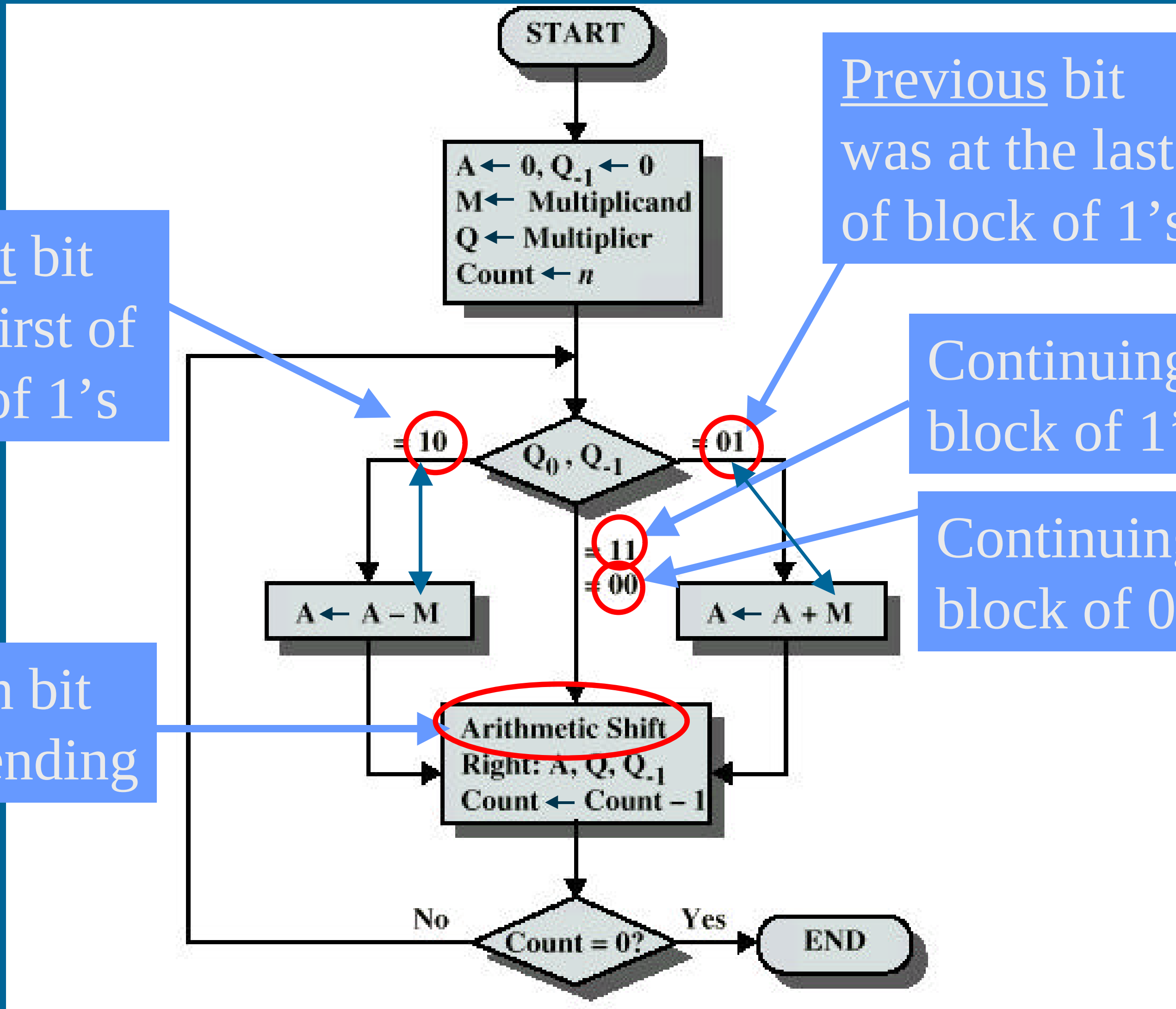
- Booth's algorithm:
  - combine all adjacent 1's in multiplicand together, replace all additions by one subtraction and one addition (to result)

5 * 7 ⇨ 0101 * 0111

⇨ 0101 * (-0001 +1000) ⇨

```
+0101000
   - 0101
=  100011
```

Booth's Algorithm (5)



Current bit is the first of block of 1's

Previous bit was at the last of block of 1's

Continuing block of 1's

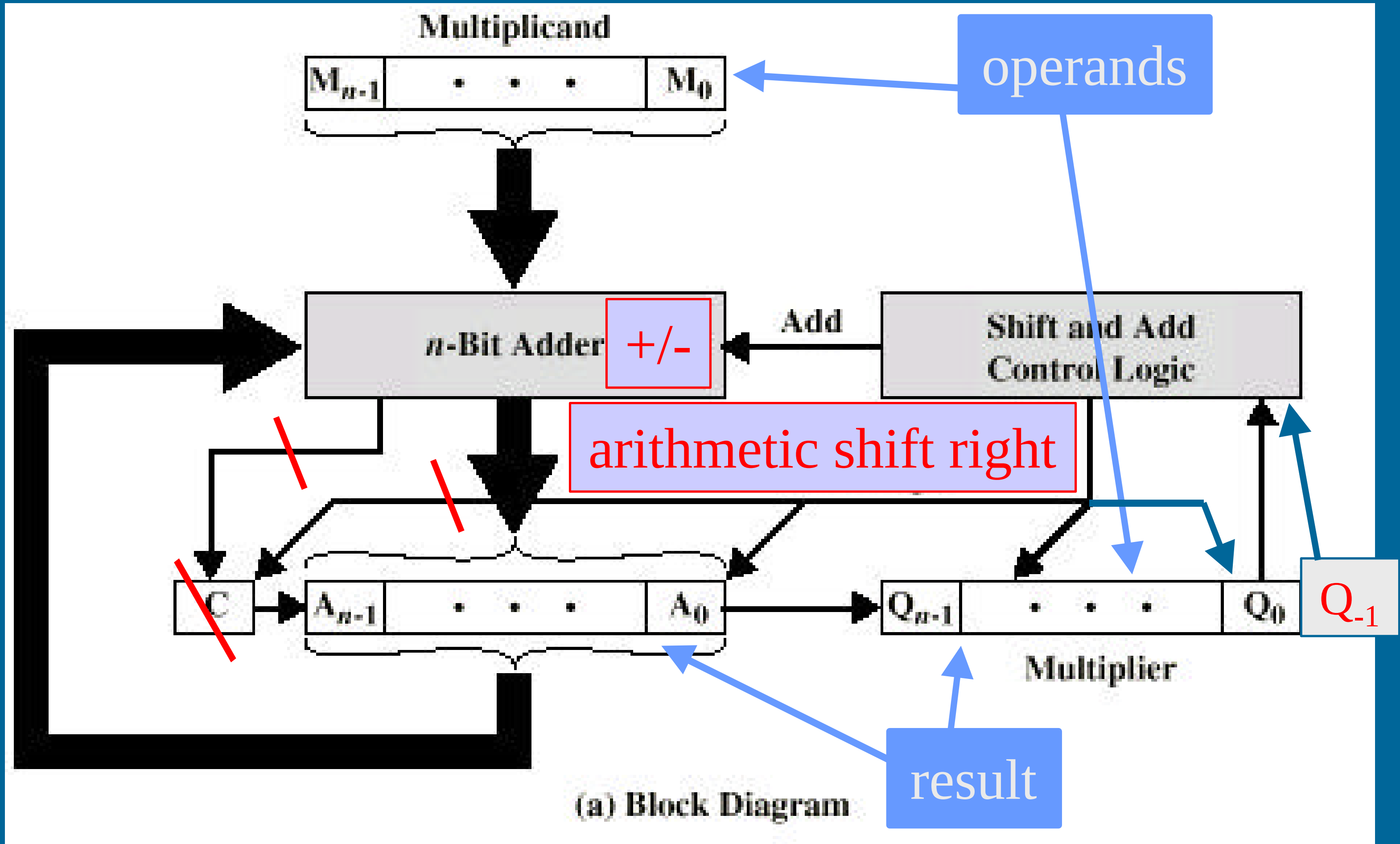
Continuing block of 0's

Sign bit extending

(Fig. 8.12)

Booth's Algorithm for Twos Complement Multiplication

Fig. 8.12

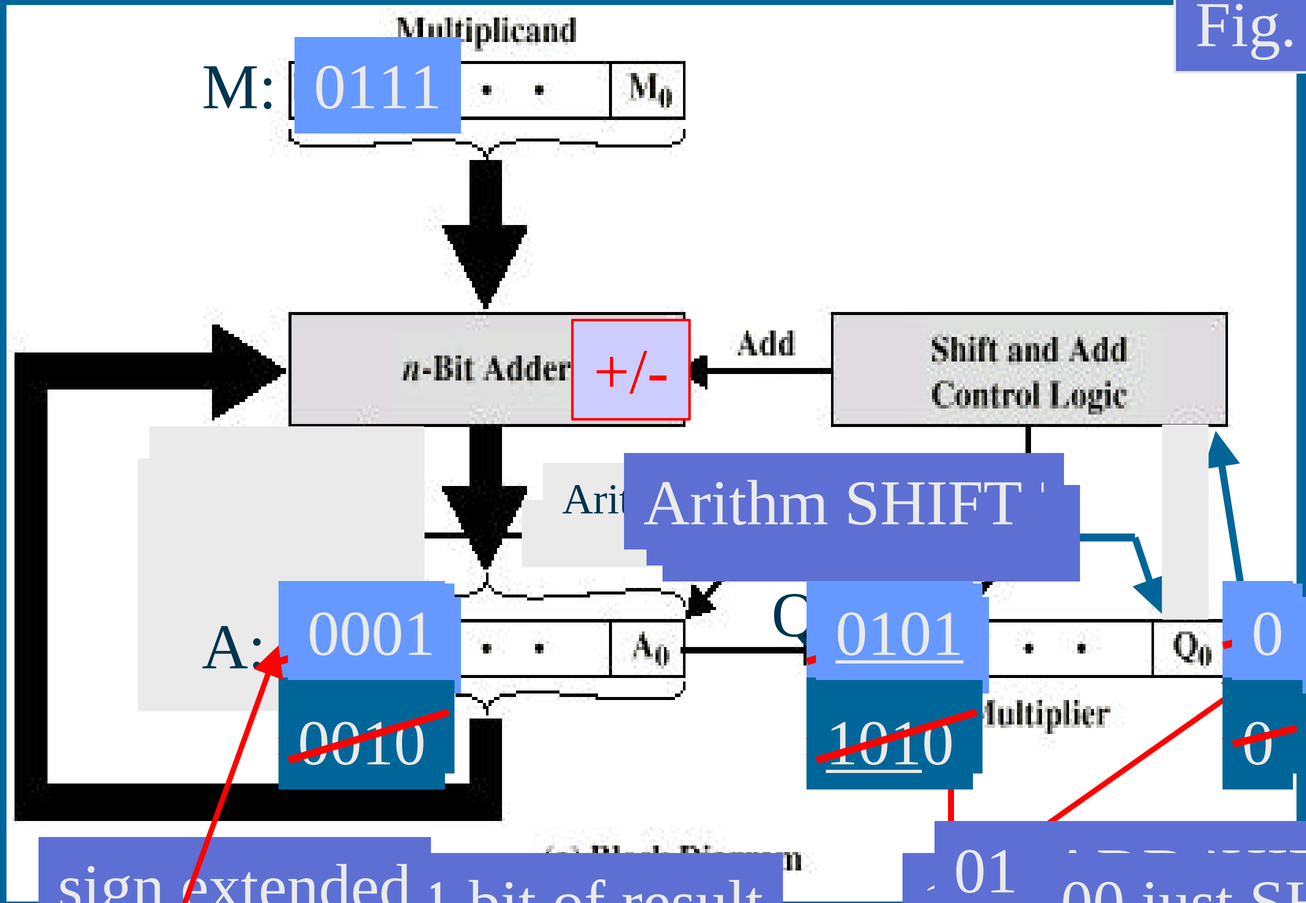


Booth's Algorithm Example (15)

7 * 3 = ?

= 0001 0101 = 21

Fig. 8.12



sign extended

1 bit of result

Carry bit was lost

01

00 just SHIFT

Integer Division

- Like in school algorithm Fig. 8.15
  - easy: new quotient digit 0 or 1
  - M register for dividend (jaettava)
  - Q register for divisor & quotient (jakaja, osamäärä)
  - A register for (partial) remaind (jakojäännös)

Floating Point Representation

-0.000 000 000 123 = -1.23 * 10^{-10}

+0.123 = +1.23 * 10^{-1}

+123.0 = +1.23 * 10^{2}

+123 000 000 000 000 = +1.23 * 10^{14}

"+"	"14"	"1.23"

sign exponent mantissa or significand

(exponentti) (mantissa)

IEEE 32-bit Floating Point Standard

IEEE Standard 754

“+”	“14”	“1.1875” = “1.0011”
sign	exponent	mantissa or significand

- 1 bit for sign, 1 $\Rightarrow$ “-”, 0 $\Rightarrow$ “+”
- I.e., Stored value S $\Rightarrow$ Sign value = $(-1)^S$

IEEE 32-bit FP Standard

"+"	"15"	"1.1875" = "1.0011"
sign	exponent	mantissa or significand

- 8 bits for exponent, 2^8-1= 127 biased form

exponent = 5 $\xrightarrow{\text{store}}$ 5+127 = 132 =1000 0100

exponent = -1 $\xrightarrow{\text{store}}$ -1+127 = 126 = 0111 1110

exponent = 0 $\xrightarrow{\text{store}}$ 0+127 = 127 = 0111 1111

– stored exponents 0 and 255 are special cases
  - stored range: **1 - 254** $\Rightarrow$ true range: **-126 - 127**

IEEE 32-bit FP Standard (7)

1/8 = 0.1250
1/16= 0.0625
0.1875

“+”	“15”	“0.1875” = “0.0011”
sign	exponent	mantissa or significand

- 23 bits for mantissa, stored so that

1) Binary point (.) is assumed just right of first digit

2) Mantissa is <u>normalised</u>, so that leftmost digit is 1

3) Leftmost (most significant) digit (1) is not stored (implied bit)

mantissa	exponent
0.0011	“15”
1.100	“12”
1000	“12”

24 bit mantissa!

IEEE 32-bit FP Values

$23 = +10111.0 * 2^0 = +1.0111 * 2^4 = ?$

4+127=131

sign	exponent	mantissa or significand
0	1000 0011	011 1000 0000 0000 0000 0000
1 bit	8 bits	23 bits

$1.0 = +1.0000 * 2^0 = ?$

0+127 = 127

sign	exponent	mantissa or significand
0	0111 1111	000 0000 0000 0000 0000 0000
1 bit	8 bits	23 bits

IEEE 32-bit FP Values

0	1000 0000	111 1000 0000 0000 0000 0000
sign 1 bit	exponent 8 bits	mantissa or significand 23 bits

X = ?

$X = (-1)^0 * 1.1111 * 2^{(128-127)}$

$= 1.1111_2 * 2$

$= (1 + 1/2 + 1/4 + 1/8 + 1/16) * 2$

$= (1 + 0.5 + 0.25 + 0.125 + 0.0625) * 2$

$= 1.9375 * 2$

$= 3.875$

IEEE-754 Floating-Point Conversion

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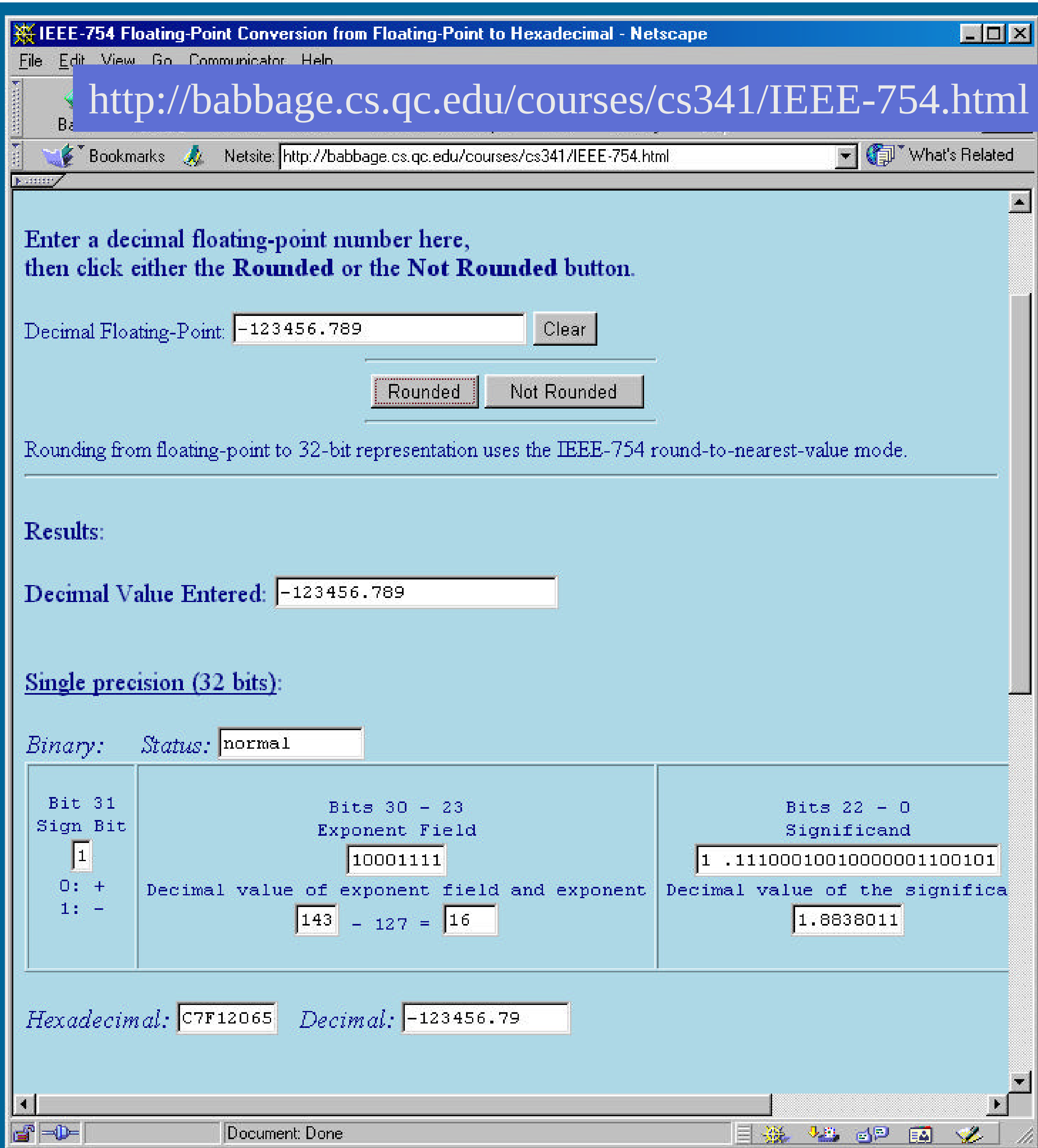
(The City University of New York)

http://babbage.cs.qc.edu/courses/cs341/IEEE-754.html



IEEE FP Standard

- Single Precision (SP) 32 bits
- Double Precision (DP) 64 bits

(yksin- ja kaksinkertainen tarkkuus)

Table 8.3

- Special values
  - -0, $+\infty$, $-\infty$, NaN
  - denormalized values

Table 8.4

Not a Number

IEEE SP FP Range

- Range
  - 8 bit exponent, effective range: -126 … +127
  - range $2^{-126} \ldots 2^{127} \approx -10^{-38} \ldots 10^{38}$
- Accuracy
  - 23 bit mantissa, 24 bit effective mantissa
  - change least significant digit in mantissa?
  - $2^{24} \approx 1.7 * 10^{-7} \approx$ 6 decimal digits

Floating Point Arithmetic (4)

Table 8.5

- Relatively simple
- Done from registers with all bits
  - implied bit included
- Add/subtract
  - more complex than multiplication
  - denormalize first one operand so that both have same exponent
- Multiplication/Division
  - handle mantissa and exponent separately

FP Add or Subtract (4)

- Check for zeroes — $1.234 \bullet 10^4$ + $4.444 \bullet 10^6$
  - trivial if one or both operands zero
- Align mantissas — $0.01234 \bullet 10^6$ $4.444 \bullet 10^6$
  - same exponent
- Add/subtract — $4.45634 \bullet 10^6$
  - carry?
    $\Rightarrow$ shift right and add increase exponent
- Normalize result — $4.45634 \bullet 10^6$
  - shift left, reduce exponent

FP Special Cases

- Exponent overflow (ylivuoto)
  - above max — Exception Or $\pm\infty$?
- Exponent underflow (alivuoto)
  - below min — Exception or zero?
- Mantissa (significant) underflow
  - in denormalizing may move bits too much right
  - all significant bits lost? — Oooops, lost data!
- Mantissa (significant) overflow — Fix it
  - result of adding mantissas may have carry

FP Multiplication (Division) (7)

Check for zeroes

Result 0, ±∞ ??

Add exponents

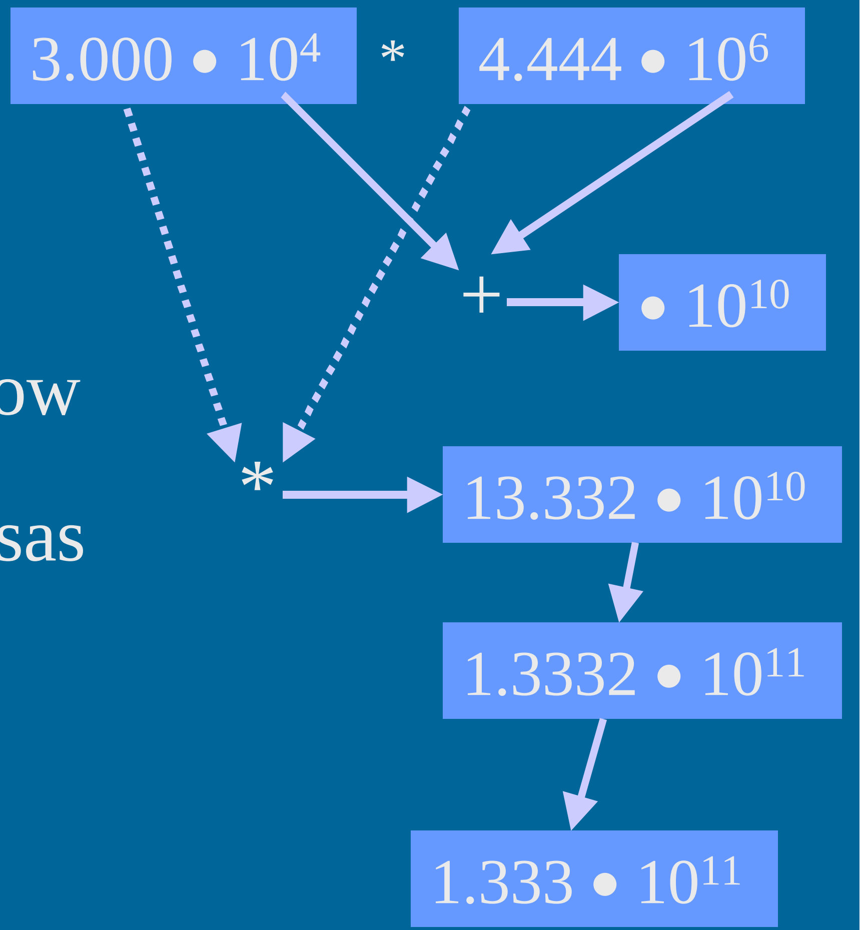
Subtract extra bias

Report overflow/underflow

Multiply (divide) mantissas

Normalise

Round (pyöristä)



Rounding (4)

- Guard bits
  - extra padding with zeroes
  - used with computations only
  - computations with more accuracy than data

$4.444 \bullet 10^6$

$4.444\underline{00} \bullet 10^6$

$2.0 - 1.9999 \approx 1.000000 \bullet 2^1 - 0.1111111 \bullet 2^1$

$= 1.000000 \bullet 2^1 - 1.111111 \bullet 2^0$

normalised

6 bit mantissa

$$
\begin{array}{r}
1.000000 \bullet 2^1 \\
-\ 0.111111 \bullet 2^1 \\
\hline
= 0.000001 \bullet 2^1 \\
= 1.000000 \bullet 2^{-5}
\end{array}
$$

Different accuracy!

Align mantis-sas

$$
\begin{array}{r}
1.000000\ 00 \bullet 2^1 \\
-\ 0.111111\ 10 \bullet 2^1 \\
\hline
= 0.000000\ 10 \bullet 2^1 \\
= 1.000000\ 00 \bullet 2^{-6}
\end{array}
$$

2 guard bits

Rounding Choices (4)

4 digit accuracy in memory?	3.1234 or -4.5678
• Nearest representable	3.123 or -4.568
• Toward +∞	3.124 or -4.567
• Toward -∞	3.123 or -4.568
• Toward 0	3.123 or -4.567

IEEE ∞ and NaN

- ∞
  - outside range of finite numbers
  - rules for arithmetic with ∞
- NaN
  - invalid operation (E.g., 0.0/0.0) can result to NaN or exception
    - user control
    - quiet NaN instead of exception

Table 8.6

IEEE Denormalized Numbers (4)

- Problem: What to do when can not normalize any more?
  - Exponent would underflow
- Answer: Denormalized representation
  - smallest representable exponent reserved for this purpose
  - mantissa is not normalized
  - smallest (closest to zero) value is now much smaller than with normalized representation

$0.003456 \bullet 10^{-99}$

6 decimal digit mantissa

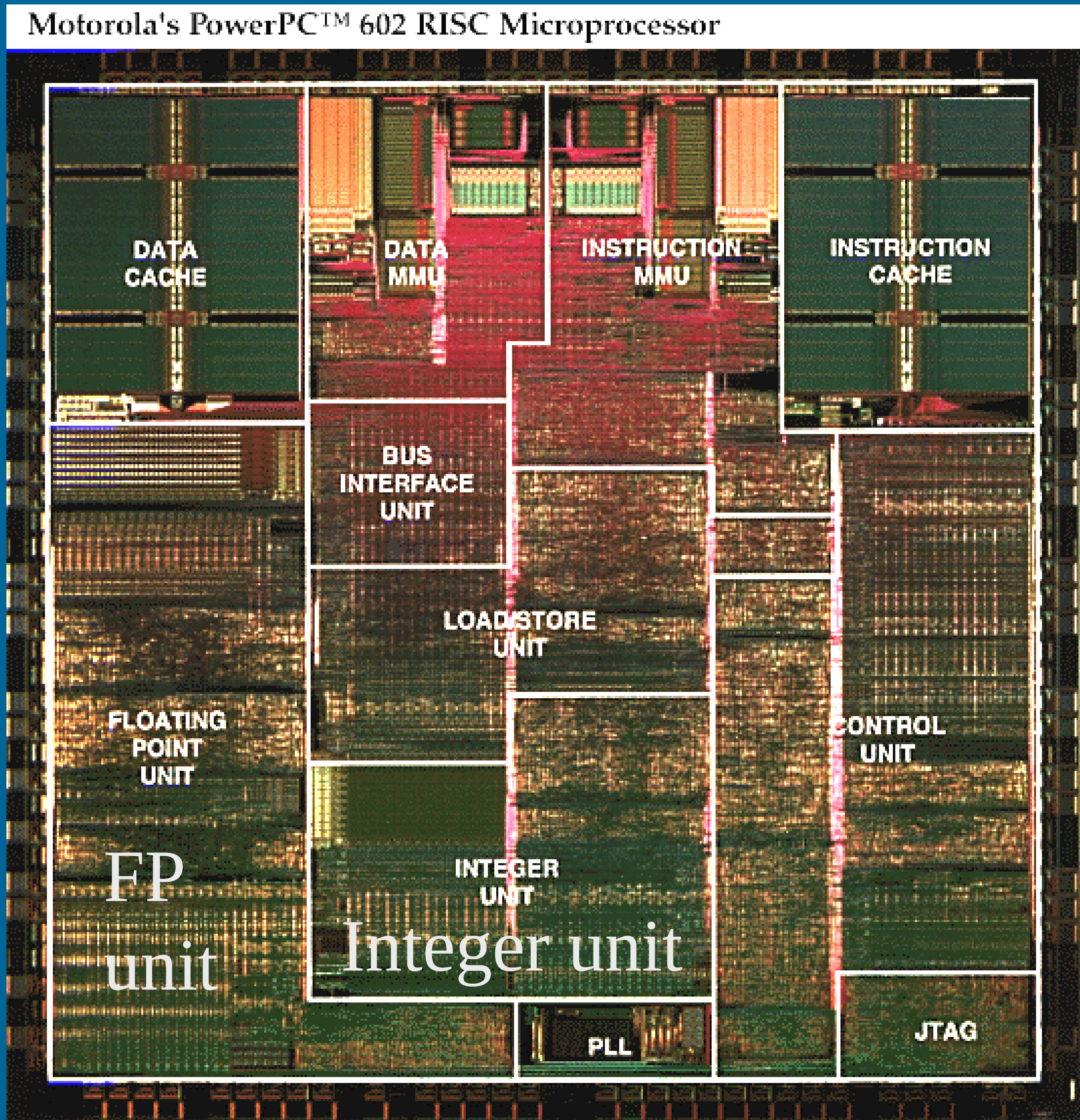
Smallest representable exponent

$1.000000 \bullet 10^{-99}$

?

$0.000001 \bullet 10^{-99}$

-- End of Chapter 8: Arithmetic --



http://infopad.eecs.berkeley.edu/CIC/die_photos/