Digital Logic
Appendix A

Boolean Algebra
Gates
Combinatorial Circuits
Sequential Circuits

Boolean Algebra

- George Boole
  - ideas 1854
- Claude Shannon,
  - apply to circuit design, 1938 (piirisuunnittelu)
- Describe digital circuitry function
  - programming language?
- Optimise given circuitry
  - use algebra (Boolean algebra) to manipulate (Boolean) expressions into simpler expressions

Boolean Algebra

- Variables: A, B, C
- Values: TRUE (1), FALSE (0)
- Basic logical operations:
  - binary: AND (•), OR (+) $A \bullet B = AB$ $B + C$
  - unary: NOT (¯) $\overline{A}$ (ja, tai, ei)
- Composite operations, equations
  - precedence: NOT, AND, OR
  - parenthesis $D = A + \overline{B} \bullet C = A + ((\overline{B})C)$

Boolean Algebra

- Other operations
  - NAND
  - NOR
- Truth tables
  - What is the result of the operation?

$A \text{ NAND } B = \text{NOT}(A \text{ AND } B) = \overline{AB}$

$A \text{ NOR } B = \text{NOT}(A \text{ OR } B) = \overline{A + B}$

Table A.1

P \ Q	AND	0	1
	0	0	0
	1	0	1

Postulates, Identities in Boolean Algebra

- How can I manipulate expressions?
  - Simple set of rules?
- Basic identities — Table A.2
  - commutative laws (vaihdantalait)
  - distributive laws (osittelulait)
  - identity elements (identiteetit)
  - inverse elements (vasta-alkiot)
  - associative laws (liitäntälait)
  - DeMorgan's theorem (DeMorganin laki)

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Gates (portit)

- Fundamental building blocks
  - easy to build
  - implement basic Boolean algebra operations
- Combine to build more complex circuits
  - memory, adder, multiplier (yhteenlaskupiiri, kertolaskupiiri)
- 1-3 inputs, 1 output

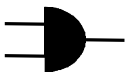

Fig. A.1 AND

- Gate delay (viive)
  - change inputs, new output available after gate delay

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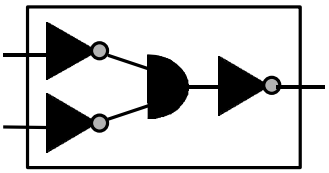
Functionally Complete Set

- Can build all basic gates (and, or, not) from a smaller set of gates
  - and, or, not (trivial!)
  - and, not
    - OR?

$$A + B = \overline{\overline{A} \bullet \overline{B}}$$



  - or, not
  - nand
  - nor

Fig A.2

Fig A.3

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Combinational Circuits (3)

(yhdistelmä-piirit)

- Interconnected set of gates
  - change input, wait for gate delays, new output
- Output is Boolean function of input signals
  - m (binary, Boolean) inputs
  - n (binary, Boolean) outputs
- Described in three ways
  - Boolean equations (one for each output)

$$F = \overline{A}B\overline{C} + \overline{A}BC + AB\overline{C}$$

  - truth table

Table A.3

  - graphical symbols describe implementation with gates and wires

Fig A.4

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Simplify Presentation (and Implementation) (3)

- Boolean equations
  - Sum of products form (SOP)

$$F = \overline{A}B\overline{C} + \overline{A}BC + AB\overline{C}$$

Fig A.4

Boolean algebra

  - Product of sums form (POS)

$$F = (A+B+C)\bullet(A+B+\overline{C})\bullet(\overline{A}+B+C)\bullet(\overline{A}+B+\overline{C})\bullet(\overline{A}+\overline{B}+\overline{C})$$

Fig A.5

Which presentation is better?
Fewer gates? Smaller area on chip?
Smaller circuit delay? Faster?

Algebraic Simplification

- Circuits become too large to handle?
- Use basic identities to simplify Boolean expressions

$$F = \overline{A}B\overline{C} + \overline{A}BC + AB\overline{C}$$
$$= \overline{A}B + B\overline{C} = B(\overline{A}+\overline{C})$$

Fig A.5 → Fig A.6

- May be difficult to do
- How to do it automatically?
- Build a program to do it “best”?

Karnaugh Map Squares

- Each square represents complete input value combination (canonical form)
  - adjacent squares differ only in <u>one</u> input value (wrap around)

order!!

CD: / AB	**00**	**01**	**11**	**10**
00	0000	0001	0011	0010
01	0100	0101	0111	0110
11	1100	1101	1111	1110
10	1000	1001	1011	1010

Square for <u>input value</u> combination

$\overline{A}\overline{B}CD = 1001$

Karnaugh Maps

- Represent Boolean function (I.e., circuit) <u>truth table</u> in another way

Fig A.7

  - each square is **one product** in sums-of-products (SOP) presentation
  - value is one (1) iff corresponding input values give value 1
  - value is function value for those input values

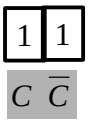
CD

AB	**00**	**01**	**11**	**10**
00			1	
01				
11	1			
10		1		

$F = \overline{A}\overline{B}CD + AB\overline{C}\overline{D} + \overline{A}\overline{B}C\overline{D}$

Karnaugh Map Simplification

- Starting point:
  - Adjacent squares differ only in one input variable value



Adjacent squares have value 1

↓

Input values differ only in one variable

↓

Value of that variable is irrelevant
(when all other input variables are fixed
to corresponding values for those squares)

↓

Can ignore that variable for those expressions

Using Karnaugh Maps to Minimize Boolean Functions (6)

Original function

$$f = \bar{a}\bar{b}\bar{c}d + \bar{a}\bar{b}cd + a\bar{b}c\bar{d} + a\bar{b}cd + abcd + abc\bar{d} + a\bar{b}cd + a\bar{b}c\bar{d}$$

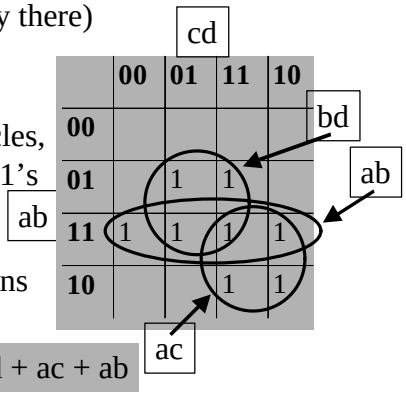
Canonical form (now already there)

Karnaugh Map

Find smallest number of circles, each with largest number of 1's

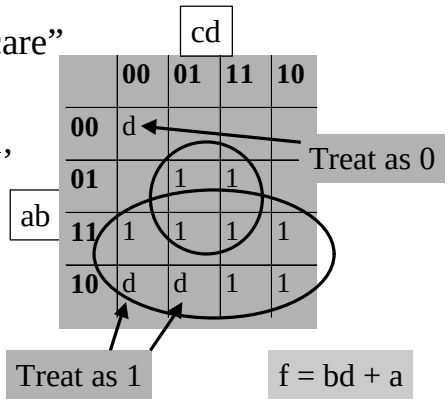
Select parameter combinations corresponding to the circles

Get reduced function f = bd + ac + ab



Impossible Input Variable Combinations

- What if some input combinations can never occur?
  - Mark them"don't care" or "d"
  - treat them as 0 or 1, whichever is best
  - more room to optimize



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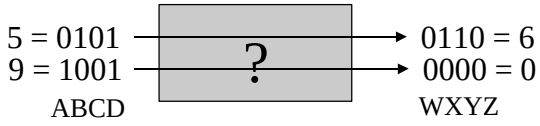
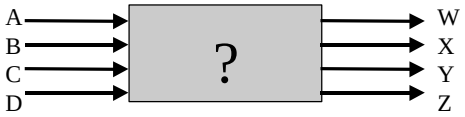
Example: Circuit to add 1 (mod 10) to 4-bit BCD decimal number (4)





Truth table? Table A.4

Karnaugh maps for W, X, Y and Z? Fig. A.10

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Quine-McKluskey Method

- Another method to minimise Boolean expressions
- Why?
  - Karnaugh maps become complex with 6 input variables
- Quine-McKluskey method
  - tabular method
  - automatically suitable for programming
  - details skipped

Basic Combinational Circuits

- Building blocks for more complex circuits
  - Multiplexer
  - Encoders/decoder
  - Read-Only-Memory
  - Adder

Multiplexers (2)

- Select one of many possible inputs to output
  - black box — Fig A.12
  - simple truth table — Tbl A.7
  - implementation — Fig A.13
- Each input "line" can be many parallel lines
  - select one of three 16 bit values — Fig A.14
    - $C_{0..15}$, $IR_{0..15}$, $ALU_{0..15}$
  - simple extension to one line selection

Encoders/Decoders

- Only one of many Encoder input or Decoder output lines can be 1
- Encode the line number as output
  - hopefully less pins (wires) needed this way
  - Example:
    - encode 8 input wires with 3 output pins
    - route 3 wires around the board
    - decode 3 wires back to 8 wires at target — Fig A.15

Read-Only-Memory (ROM) (4)

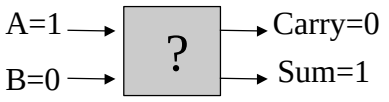
- Given input values, get output value
- Consider input as address, output as contents of memory location
- Truth tables for a ROM — Table A.8
  - 64 bit ROM
  - 16 words, each 4 bits wide

Mem (7) = 4 Mem (11) = 14

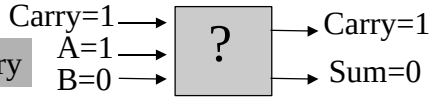
- Implementation with decoder & or gates

Adders (4)

1-bit adder



1-bit adder with carry



Implementation — Table A.9, Fig A.22

Build 4-bit adder from 4 1-bit adders — Fig A.21

Sequential Circuit

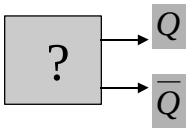
(sarjalliset piirit)

- Circuit has (modifiable) internal state
- Output of circuit depends (also) on internal state
  - not only from current inputs
  - output = f(input, state)
  - new state = f(input, state)
- Processor control, registers

Flip-Flop

(kiikku)

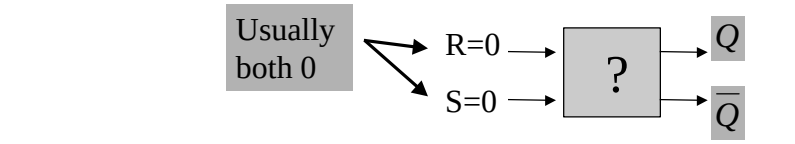
- 2 states for Q (0 or 1, true or false)
- 2 outputs Q and $\overline{Q}$
  - complement values
  - both always available on different pins



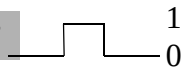
- Need to be able to change the state (Q)

S-R Flip-Flop or S-R Latch (4)

(laituri)

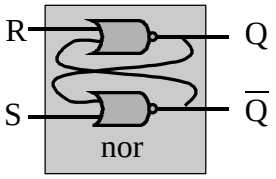


"Write 1" = "set S=1 for a short time" = "set"



"Write 0" = "set R=1 for a short time" = "reset"

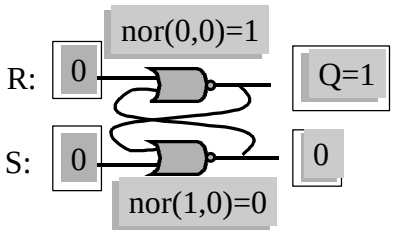
nor (0, 0) = 1
nor (0, 1) = 0
nor (1, 0) = 0
nor (1, 1) = 0



S-R Latch Stable States (3)

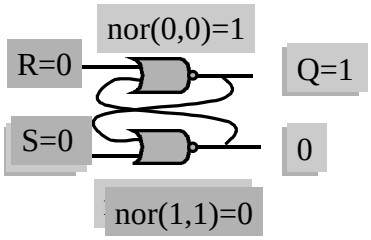
- 1 bit memory (value = value of Q)
- bi-stable, when R=S=0
  - Q=0?
  - Q=1?

nor (0, 0) = 1
nor (0, 1) = 0
nor (1, 0) = 0
nor (1, 1) = 0



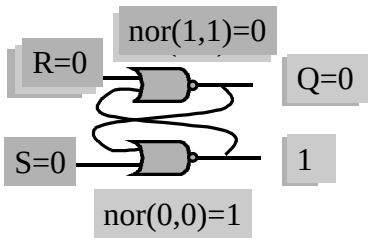
S-R Latch Set (1) and Reset (0) (n)

Write 1: S=0 → 1 → 0



Write 0: R=0 → 1 → 0

nor (0, 0) = 1
nor (0, 1) = 0
nor (1, 0) = 0
nor (1, 1) = 0



Clocked Flip-Flops

- State change can happen only when clock is 1
  - more control on state changes
- Clocked S-R Flip-Flop Fig. A.26
- D Flip-Flop Fig. A.27
  - only one input D
    - D = 1 and CLOCK → write 1
    - D = 0 and CLOCK → write 0
- J-K Flip-Flop Fig. A.28
  - Toggle Q when J=K=1 Fig. A.29

Registers (2)

- Parallel registers
  - read/write
  - CPU user registers
  - additional internal registers
- Shift Registers
  - shifts data 1 bit to the right
  - serial to parallel?
  - ALU operation?
  - rotate?

Fig. A.30

Fig. A.31

Counters (4)

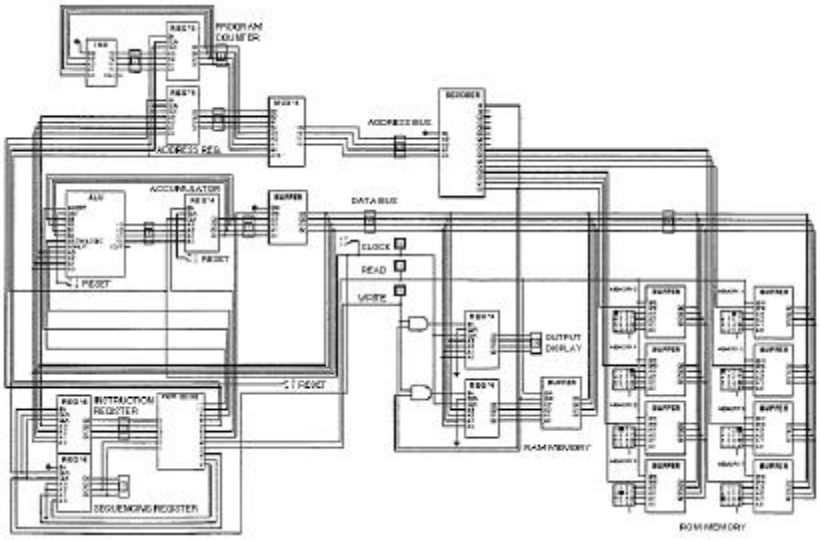
- Add 1 to stored counter value
- Counter
  - parallel register plus increment circuits
- Ripple counter
  - asynchronous
  - increment least significant bit, and handle "carry" bit as far as needed
- Synchronous counter
  - modify all counter flip-flops simultaneously
  - faster, more complex, more expensive

Summary

- Boolean Algebra → Gates → Circuits
  - can implement all with NANDs or NORs
  - simplify circuits: Karnaugh, Quine-McKluskey
- Components for CPU design
  - ROM, adder
  - multiplexer, encoder/decoder
  - flip-flop, register, shift register, counter

-- End of Appendix A: Digital Logic --

Simple processor



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