Computer Arithmetic
Ch 9

ALU
Integer Representation
Integer Arithmetic
Floating-Point Representation
Floating-Point Arithmetic

Arithmetic Logical Unit (ALU) (2)

(aritmeettis-looginen yksikkö)

- Does all "work" in CPU — Rest is management!
  - integer & floating point arithmetic's
  - copy values from one register to another
  - comparisons
  - left and right shifts
  - branch and jump address calculations
  - load/store address calculations
- Control signals from CPU control unit
  - what operation to perform and when

ALU Operations (5)

- Data from/to internal registers (latches)
  - input data may have been copied from normal registers, or it may have come from memory
  - output data may go to normal registers, or to memory

Fig. 9.1
(Fig. 8.1[Stal99])

- Wait for maximum gate delay
- Result is ready
- Result may (also) be in flags (lipuke)
- Flags may cause an interrupt

Integer Representation (8)

Everything with 0 and 1
no plus/minus signs
no decimal periods
assumed "on the right"

+32 +16 +8 +1

57 = 0011 1001 = 0x39

hexadecimal presentation

3*16 +9*1

- Unsigned integers
- Positive numbers easy
  - normal binary form
- Negative numbers
  - sign-magnitude
  - two's complement

sign bit = MSB = most significant bit

-57 = 1011 1001

-57 = 1100 0111

+1

"sign" bit

complements

Twos Complement

(kahden komplementti)

- Most used
- Have space for 8 bits?
  - use 7 bits for data and 1 bit for sign

+2 = 0000 0010
+1 = 0000 0001
0 = 0000 0000
-1 = 1111 1111
-2 = 1111 1110

  - just like in sign-magnitude or in one's complement (but presentation is different)

ones complement: -0 = 1111 1111

Why Two's Complement Presentation? (4)

- Math is easy to implement
  - subtraction becomes addition

X-Y = X + (-Y)

easy to do, simple circuit

- Have just one zero
  - comparisons to zero easy
- Easy to expand to presentation with more bits
  - simple circuit

57 = 0011 1001 = 0000 0000 0011 1001

-57 = 1100 0111 = 1111 1111 1100 0111

sign extension

Why Two's Complement Presentation? (3)

- Range with n bits: $-2^{n-1} \dots 2^{n-1} - 1$

  8 bits: $-2^7 \dots 2^7 - 1 = -128 \dots 127$
  32 bits: $-2^{31} \dots 2^{31} - 1 = -2\,147\,483\,648 \dots 2\,147\,483\,647$

- Overflow easy to recognise
  - add positive & negative: overflow not possible!
  - add 2 positive/negative numbers
    - if "sign" bit of result is different? ⇒ overflow!

```
  57 = 0011 1001
+ 80 = 0101 0000
----------------
 137 = 1000 1001
```

outside range

Why Two's Complement Presentation? (1)

- Addition easy if one or both operands negative
  - treat them all as unsigned integers

Same circuit works for both (except for overflow check)

```
 13 = 1101
 +1 = 0001
----------
 14 = 1110
```

Digits represent 4 bit unsigned numbers

```
 -3 = 1101
 +1 = 0001
----------
 -2 = 1110
```

Digits represent 4 bit two's complement numbers

+3 = 0011

```
1100
  +1
----
1101
```


Integer Arithmetic Operations

- Negation X = -Y
- Addition X = Y+Z
- Subtraction X = Y-Z
- Multiplication X = Y*Z
- Division X = Y / Z

Integer Negation (3)

- Step 1: negate all bits
- Step 2: add 1
  - Step 3: special cases
    - ignore carry bit
      - negate 0?
    - check that sign bit really changes
      - can not negate smallest negative
      - results in exception

```
57 = 0011 1001
     1100 0110
            +1
     1100 0111
```

```
 0 = 0000 0000
     1111 1111
            +1
-0 = 1 0000 0000
```

```
-128 = 1000 0000
bitwise not: 0111 1111
add 1: 1000 0000
```


Integer Addition and Subtraction

- Normal binary addition
  - 32 bit full adder?
- Ignore carry & monitor sign bit for overflow
- In case of SUB, complement 2nd operand
- 2 circuits
  - addition
  - complement

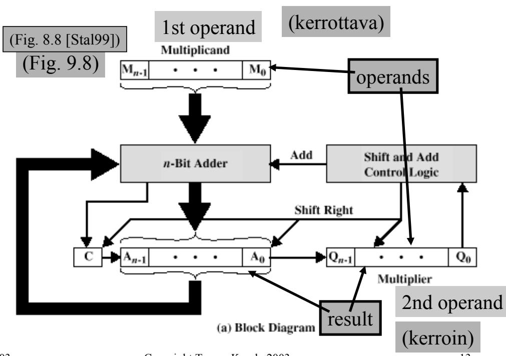
Fig. 9.6 (Fig. 8.6 [Stal99])

Integer Multiplication (4)

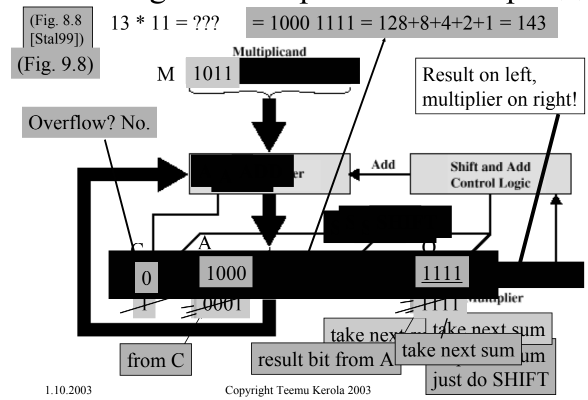
- Complex
- Operands 32 bits ⇒ result 64 bits
- "Just like" you learned at school
  - optimised for binary data
    - it is easy to multiply with 0 or 1!
- Simpler case with unsigned numbers
  - simple circuits
    - adder
    - shifter
    - wires

Fig. 9.7 (Fig. 8.7 [Stal99])

Unsigned Multiplication Example



Unsigned Multiplication Example (19)



Multiplication with Negative Values

- Multiplication for unsigned numbers does not work for negative numbers
  - algorithm applies only for unsigned integer representation
  - not the same case as with addition
- Could do it all with unsigned values
  - (a) change operands to positive values
  - (b) do multiplication with positive values
  - (c) negate result if needed
  - OK, but can do better, I.e., faster

The Gist in Booth's Algorithm (4)

Unsigned multiplication:
addition for every "1" bit in multiplier

5 * 7 ⇨ 0101 * 0111 ⇨

```
  0101
+ 01010
+ 010100
= 100011
```

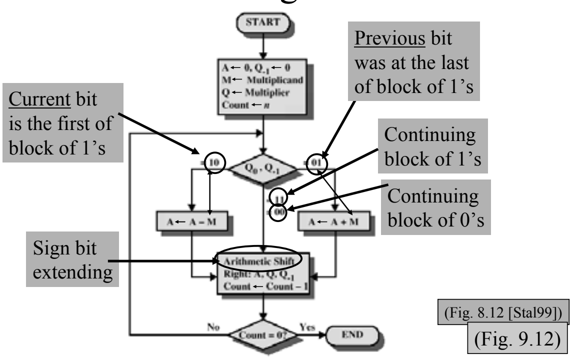
- Booth's algorithm:
  - combine all adjacent 1's in multiplier together, replace all additions by one subtraction and one addition (to result)

5 * 7 ⇨ 0101 * 0111

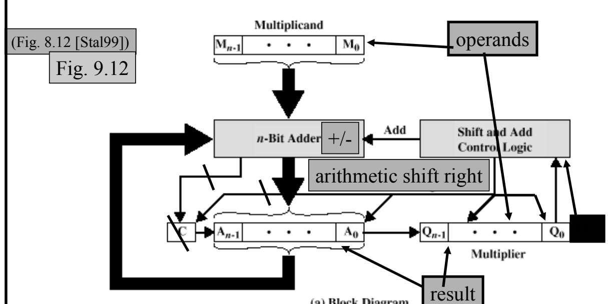
⇨ 0101 * (-0001 +1000) ⇨

```
+0101000
  - 0101
=  100011
```


Booth's Algorithm (5)



Booth's Algorithm for Twos Complement Multiplication



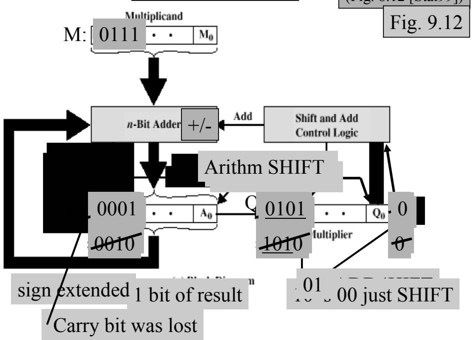
Booth's Algorithm Example (15)

7 * 3 = ?

= 0001 0101 = 21

(Fig. 8.12 [Stal99])

Fig. 9.12



Integer Division

(Fig. 8.15 [Stal99])

Fig. 9.15

- Like in school algorithm
  - easy: new quotient digit 0 or 1
  - M register for dividend (jaettava)
  - Q register for divisor & quotient (jakaja, osamäärä)
  - A register for (partial) remainder (jakojäännös)

Floating Point Representation

$-0.000\ 000\ 000\ 123 = -1.23 * 10^{-10}$

$+0.123 = +1.23 * 10^{-1}$

$+123.0 = +1.23 * 10^{2}$

$+123\ 000\ 000\ 000\ 000 = +1.23 * 10^{14}$

"+"	"14"	"1.23"
sign	exponent (exponentti)	mantissa or significand (mantissa)

IEEE 32-bit Floating Point Standard

IEEE Standard 754

"+"	"14"	"1.1875" = "1.0011"
sign	exponent	mantissa or significand

- 1 bit for sign, 1⇒ "-", 0 ⇒ "+"
- I.e., Stored value S ⇒ Sign value = $(-1)^S$

IEEE 32-bit FP Standard

"+"	"15"	"1.1875" = "1.0011"
sign	exponent	mantissa or significand

- 8 bits for exponent, $2^{8-1}-1 = 127$ <u>biased form</u>

exponent = 5 —store→ 5+127 = 132 =1000 0100

exponent = -1 —store→ -1+127 = 126 = 0111 1110

exponent = 0 —store→ 0+127 = 127 = 0111 1111

  - stored exponents 0 and 255 are special cases
    - stored range: **1 - 254** ⇒ true range: **-126 - 127**

IEEE 32-bit FP Standard (7)

"+"	"15"	"0.1875" = "0.0011"
sign	exponent	mantissa or significand

1/8 = 0.1250
1/16 = 0.0625
0.1875

- 23 bits for mantissa, stored so that

1) Binary point (.) is assumed just right of first digit

2) Mantissa is normalised, so that leftmost digit is 1

3) Leftmost (most significant) digit (1) is not stored (implied bit)

mantissa	exponent
0.0011	"15"
1.100	"12"
1000	"12"

24 bit mantissa!

IEEE 32-bit FP Values

$23.0 = +10111.0 * 2^0 = +1.0111 * 2^4 = ?$

4+127=131

0	1000 0011	011 1000 0000 0000 0000 0000
sign 1 bit	exponent 8 bits	mantissa or significand 23 bits

$1.0 = +1.0000 * 2^0 = ?$

0+127 = 127

0	0111 1111	000 0000 0000 0000 0000 0000
sign 1 bit	exponent 8 bits	mantissa or significand 23 bits

IEEE 32-bit FP Values

0	1000 0000	111 1000 0000 0000 0000 0000
sign 1 bit	exponent 8 bits	mantissa or significand 23 bits

X = ?

$X = (-1)^0 * 1.1111 * 2^{(128-127)}$

$= 1.1111_2 * 2$

$= (1 + 1/2 + 1/4 + 1/8 + 1/16) * 2$

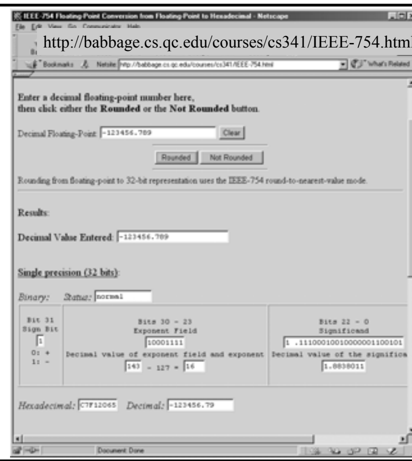
$= (1 + 0.5 + 0.25 + 0.125 + 0.0625) * 2$

$= 1.9375 * 2$

$= 3.875$

IEEE-754 Floating-Point Conversion

Christopher Vickery
Computer Science Department at Queens College of CUNY
(The City University of New York)



IEEE FP Standard

- Single Precision (SP) 32 bits
- Double Precision (DP) 64 bits

(yksin- ja kaksinkertainen tarkkuus)

Table 9.3 (Tbl. 8.3 [Stal99])

- Special values
  - -0, +∞, –∞, NaN
  - denormalized values

Table 9.4 (Tbl. 8.4 [Stal99])

Not a Number

IEEE SP FP Range

- Range
  - 8 bit exponent, effective range: -126 … +127
  - range $2^{-126} \ldots 2^{127} \approx -10^{-38} \ldots 10^{38}$
- Accuracy
  - 23 bit mantissa, 24 bit effective mantissa
  - (much) less with denormalized numbers
  - change least significant digit in mantissa?
  - $2^{24} \approx 1.7 * 10^{-7} \approx 6$ decimal digits

Floating Point Arithmetic (4)

- Relatively simple — Table 9.5 (Tbl. 8.5 [Stal99])
- Done from internal registers with all bits present
  - implied bit included
- Add/subtract
  - more complex than multiplication
  - denormalize first one operand so that both have same exponent
- Multiplication/Division
  - handle mantissa and exponent separately

FP Add or Subtract (4)

- Check for zeroes — $1.234 \bullet 10^4$ + $4.444 \bullet 10^6$
  - trivial if one or both operands zero
- Align mantissas — $0.01234 \bullet 10^6$ $4.444 \bullet 10^6$
  - same exponent
- Add/subtract — $4.45634 \bullet 10^6$
  - carry?
    ⇒ shift right and add increase exponent
- Normalize result — $4.45634 \bullet 10^6$
  - shift left, reduce exponent

FP Special Cases

- Exponent overflow (ylivuoto)
  - above max — Exception Or ±∞ ?
- Exponent underflow (alivuoto)
  - below min — Exception or zero or denormalized?
- Mantissa (significant) underflow
  - in denormalizing may move bits to the right so much that will lose significant accuracy
  - all significant bits lost? — Oooops, lost data!
- Mantissa (significant) overflow
  - result of adding mantissas may have carry — Fix it

FP Multiplication (Division) (7)

Check for zeroes
Result 0, ±∞ ?? — $3.000 \bullet 10^4$ * $4.444 \bullet 10^6$

Add exponents — + → $\bullet 10^{10}$
Subtract extra bias
Report overflow/underflow

Multiply (divide) mantissas — * → $13.332 \bullet 10^{10}$

Normalise — $1.3332 \bullet 10^{11}$

Round (pyöristä) — $1.333 \bullet 10^{11}$

Guard Bits for Better Accuracy (5)

- Guard bits — $4.444 \bullet 10^6$
  - extra padding with zeroes (before alignment)
  - used with computations only — $4.444\underline{00} \bullet 10^6$
  - computations with more accuracy than data

$2.0 - 1.984375 = 1.000000 \bullet 2^1 - 0.1111111 \bullet 2^1$
$(= 0.015625) = 1.000000 \bullet 2^1 - 1.111111 \bullet 2^0$ normalised

Align mantis-sas

6 bit mantissa:
$1.000000 \bullet 2^1$
$- 0.111111 \bullet 2^1$
$= 0.000001 \bullet 2^1$
$= 1.000000 \bullet 2^{-5}$
$= 0.03125$

Different accuracy!

100% error!

8 bit mantissa:
$1.000000\,00 \bullet 2^1$
$- 0.111111\,10 \bullet 2^1$
$= 0.000000\,10 \bullet 2^1$
$= 1.000000\,00 \bullet 2^{-6}$
$= 0.015625$

2 guard bits

Rounding Choices (5)

4 digit accuracy in memory?	3.1234 or -4.5678
• Nearest representable	3.123 or -4.568
• Toward +∞	3.124 or -4.567
• Toward -∞	3.123 or -4.568
• Toward 0	3.123 or -4.567

Intel Itanium: support to all of them

IEEE ∞ and NaN

- ∞
  - outside range of finite numbers
  - rules for arithmetic with ∞: $\infty + \infty = \infty$, etc.
- NaN
  - invalid operation (E.g., 0.0/0.0) can result to NaN or exception
    - user control
    - quiet NaN, or exception?
  - un-initialized data?
  - programming language support?

Table 9.6
(Tbl. 8.6 [Stal99])

IEEE Denormalized Numbers

- Problem: What to do when can not normalize any more?
  - Exponent would underflow
- Answer: Denormalized representation
  - smallest representable exponent <u>reserved</u> for this purpose
  - mantissa is not normalized
  - smallest (closest to zero) value is now much smaller than with normalized representation

$0.003456 \bullet 10^{-99}$

6 decimal digit mantissa

Smallest representable exponent

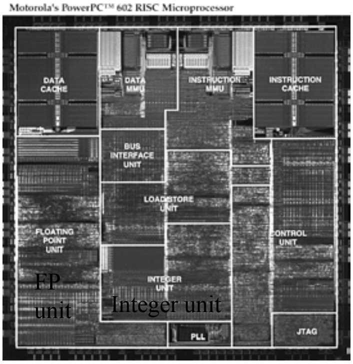
$1.000000 \bullet 10^{-99}$

?

$0.000001 \bullet 10^{-99}$

-- End of Chapter 9: Arithmetic --

Motorola's PowerPC™ 602 RISC Microprocessor



http://infopad.eecs.berkeley.edu/CIC/die_photos/