Information-Theoretic Modeling, Fall 2009

Exercises II, due Friday 25 September.

1. Prove that the logarithm function ln is strictly concave. *Hint:* Derivatives.

2. Prove Gibbs' inequality using Jensen's inequality. *Hint:* As in the slides (Lecture 3), it is sufficient to prove

$$\sum_{x \in \mathcal{X}} p(x) \ln \frac{q(x)}{p(x)} \leq 0.$$

Apply the result from Exercise 1.

3. Exercise 2.12 (item *f* optional) in Cover & Thomas (2nd edition; same as Ex. 2.16 in 1st edition).

4. What is the maximum of the entropy $H(X)$ of a random variable X with three alternative outcomes, x_1, x_2, x_3. What distribution achieves the maximum? Same for the minimum of $H(X)$.

5. There are 12 balls that look exactly the same, one of which is either heavier or lighter than the others. Your task is to determine which one of the balls is different, and whether it is heavier or lighter. To do this, you are given a balance. You can put any number of balls on each side of the balance, and the balance will tell you whether the sides are of equal weight or if the left or the right side is heavier. What is the fewest number of weighings you can use to solve the problem? Please describe your strategy and prove that it always works. If you can, also prove that the problem cannot be solved using fewer weighings.

Bonus exercise. Implement the Hamming $(7,4)$ code. The decoder should try to find a sequence of data bits d_1, d_2, d_3, d_4 for which the difference between the received seven-bit sequence and the seven-bit codeword corresponding to the data bits is minimal (at most one bit). For example, in Lecture 2, the (Hamming) distance between the received sequence 1111010 and the codeword 1011010 is one, so that the data bits is decoded as 1011. Test your code by encoding the sequence `rat.txt`. Simulate the binary symmetric channel, $p = 0.01$, by saying '`bsc.py 0.01`', then decode the noisy signal. You can create a PBM image file by the script '`txt2pbm.py`'. Compare the result with and without Hamming coding. Your experiment should be something like:

```
cat rat.txt | ./bsc.py 0.01 | ./txt2pbm.py >ratnoisy.pbm
enc <rat.txt | ./bsc.py 0.01 | dec | ./txt2pbm.py >ratham.pbm
```

where `enc` and `dec` are the encoding and decoding commands, respectively. The example file `rat.txt` and the scripts `bsc.py` and `txt2pbm.py` are available at the course web page.

Figure 1: `ratnoisy.pbm`