Information-Theoretic Modeling

Lecture 2: Noisy Channel Coding

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UNIVERSITY OF HELSINKI

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   - Reliable communication
   - Error correcting codes
   - Repetition codes

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   - Parity Check Codes
   - Hamming (7,4)

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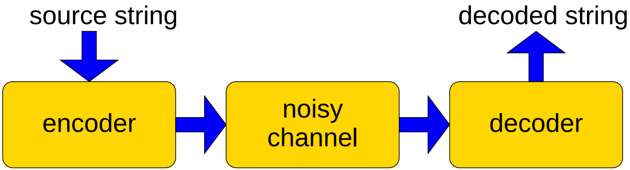
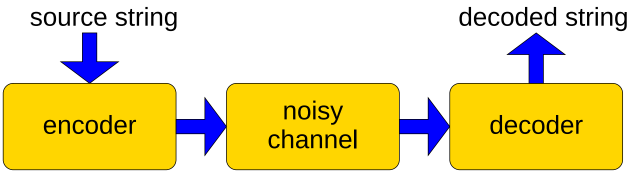
Can we recover the original message (without errors) from a noisy code string?

Error correcting codes



Error correcting codes



We want to minimize two things:

1. Length of the code string.
2. Probability of error.

Repetition codes

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Get it? Get it? Get it? Get it? Get it? Get it? Get it? G

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If errors independent and symmetric, probability of error reduced to $3(1-p)p^2 + p^3 \approx 3p^2$, where p is the error rate of the channel.

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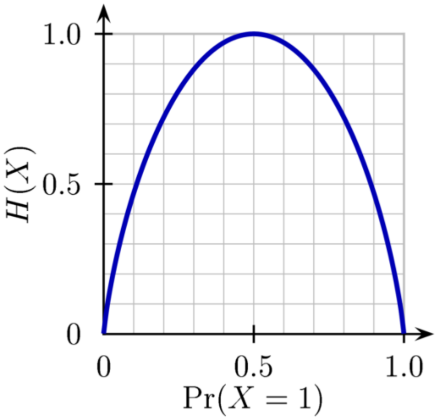
The quantity $H(p)$ is the binary *entropy*. We will get to know it very well during this course.

Channel Capacity



Noisy Channel Coding Theorem

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So what?

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Assume you want to transmit data with probability of error 10^{-15} over a BSC, $p = 0.1$. Using a repetition code, we need to make the message **63** times as long as the source string.
(Exercise: Check the math. Hint: binomial distribution.)

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Hamming Codes

Richard W. Hamming (11.2.1915–7.1.1998)

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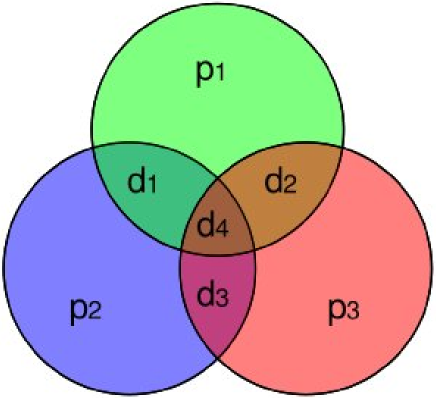
- By clever use of more than one parity bits, we can actually identify where the error occurred and thus also *correct errors*.
- Designing ways to add as few parity bits as possible to correct and detect errors is a *really* hard problem.

Hamming (7,4)



4 data bits (d_1, d_2, d_3, d_4), 3 parity bits (p_1, p_2, p_3)

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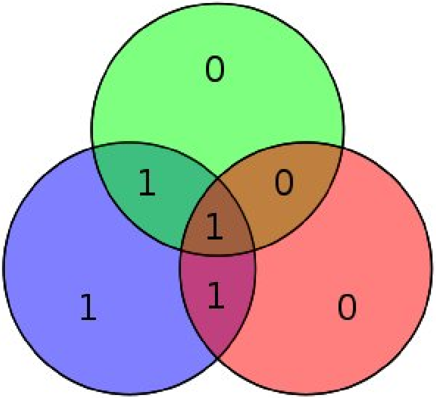
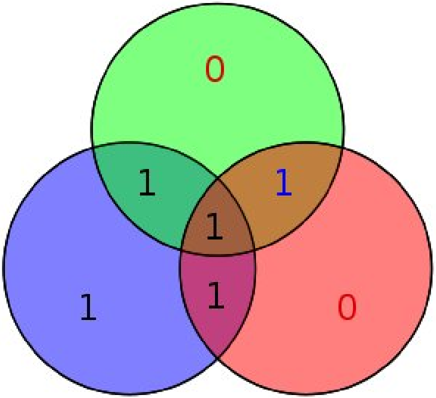


source string 1011, parity bits 010

Hamming (7,4)



error in data bit d_2 $(0 \mapsto 1)$ is identified and corrected

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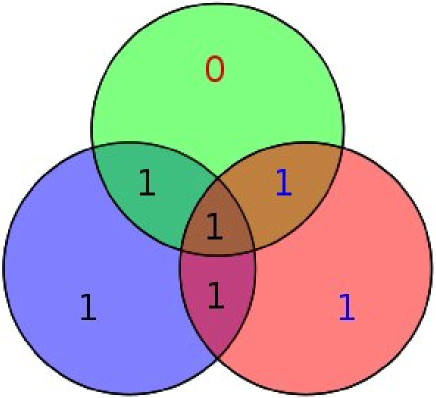

two errors can be detected but not corrected

Advanced Error Correcting Codes

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Massive research effort: At ISIT-09 conference, 12 sessions (4 talks in each) about LDPC codes.

Next Lecture

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- first exercises (questions anyone?)