Information-Theoretic Modeling

Lecture 3: Mathematical Preliminaries

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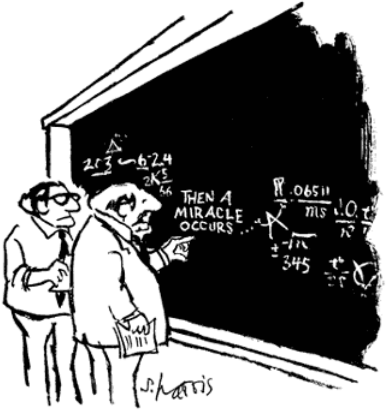

"I think you should be more explicit here in step two."

1. Calculus
   - Limits and Convergence
   - Convexity

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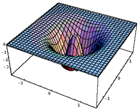
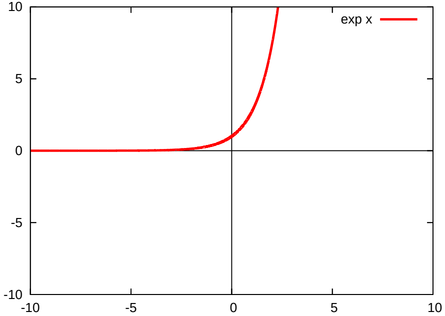
   - Limits and Convergence
   - Convexity
2. Probability
   - Probability Space and Random Variables
   - Joint and Conditional Distributions
   - Expectation
   - Law of Large Numbers
3. Inequalities
   - Jensen's Inequality
   - Gibbs's Inequality



Exponent Function



Exponent function $\exp : \mathbb{R} \to \mathbb{R}^+$, $\exp k = e^k = \overbrace{e \times e \times \ldots \times e}^{k}$:
multiplicative growth (nuclear reaction, "interest on interest", ...)

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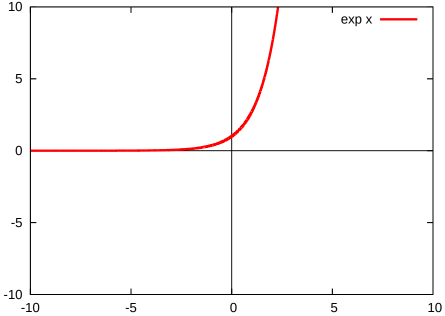

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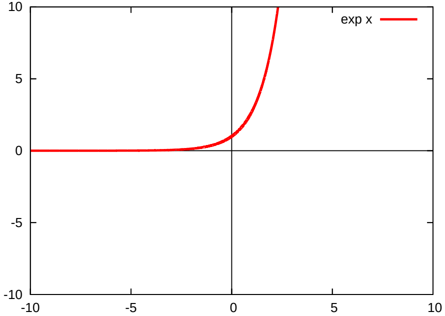

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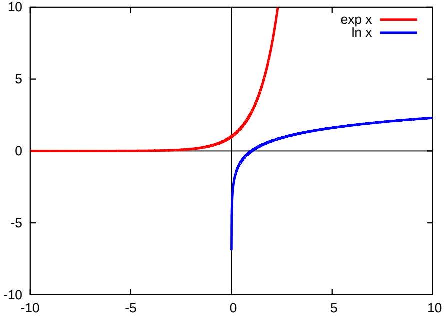
Examples: Logarithm



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time to grow to x, number of digits ($\log_{10}$).

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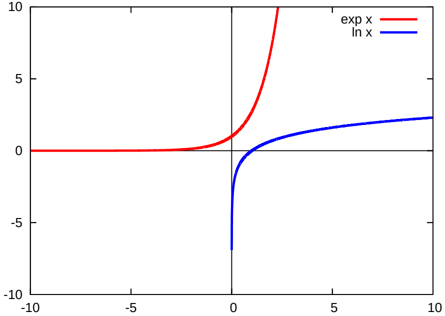

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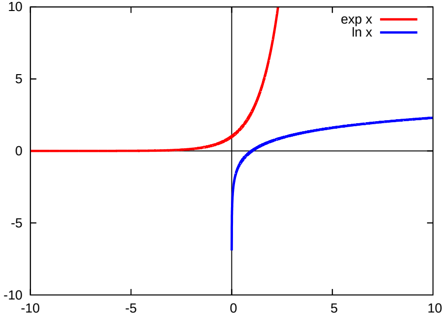
General (base a) logarithm, $\log_a a^x = x$: $$\log_a x = \frac{1}{\ln a} \ln x$$

Logarithm Function



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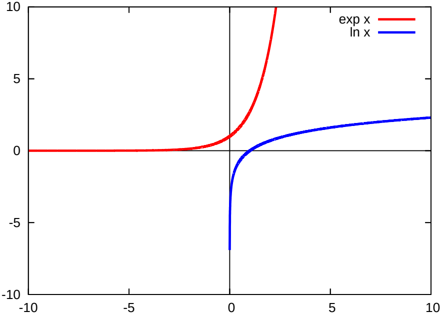

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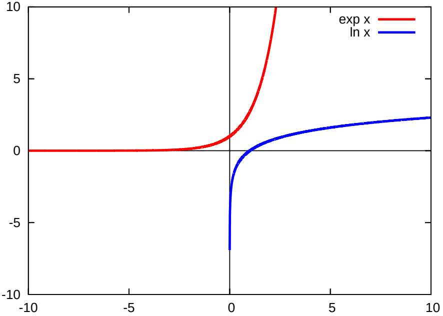

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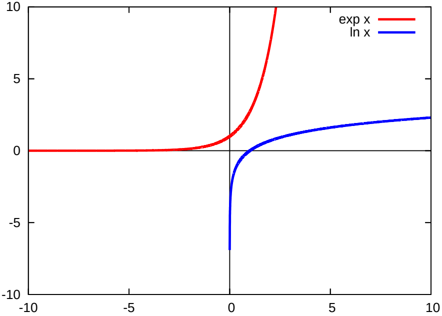

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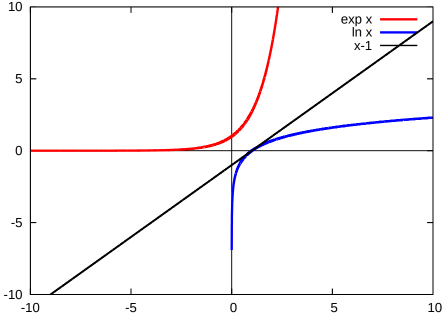

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(NB: doesn't work with $\log_a x$ if $a \neq e$)

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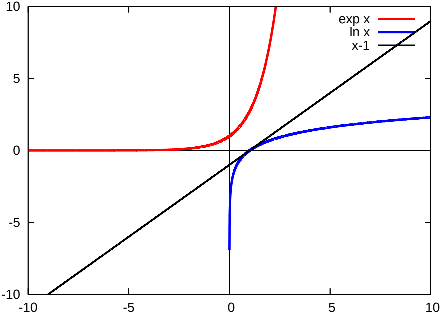

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$$\frac{d \ln x}{dx} = \frac{1}{x}$$

Limits and Convergence

- A sequence of values $(x_i \ : \ i \in \mathbb{N})$ *converges* to *limit* L, $\lim_{i\to\infty} x_i = L$, iff for any $\epsilon > 0$ there exists a number $N \in \mathbb{N}$ such that

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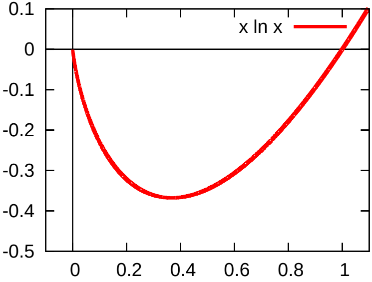
- $f(x)$ has a *limit* L as x approaches c, $\lim_{x\to c} f(x) = L$, (from above c^+/below c^-) iff for any $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{for all} \begin{cases} c < x < c + \delta & \text{`above'} \\ c - \delta < x < c & \text{'below'} \\ 0 < |x - c| < \delta & \text{---} \end{cases}$$

Example: Logarithm Again



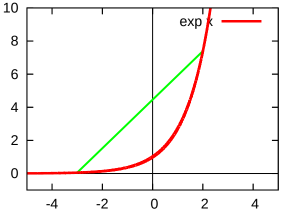
Even though $x \ln x$ is undefined at $x = 0$, we have (by l'Hôpital's rule):

$$\lim_{x \to 0^+} x \ln x = 0 \quad .$$

Convexity

Function $f : \mathcal{X} \to \mathbb{R}$ is said to be **convex** iff for any $x, y \in \mathcal{X}$ and any $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \ .$$



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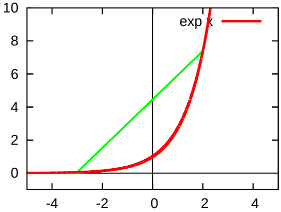

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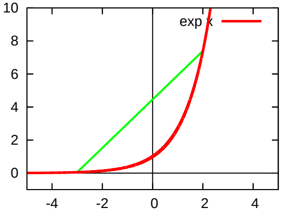

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Function f is (strictly) **concave** iff the above holds for $-f$.

Convexity and Derivatives

Theorem

If function f has a second derivative f'', and f'' is non-negative (≥ 0) for all x, then f is convex. If f'' is positive (> 0) for all x, then f is *strictly* convex.

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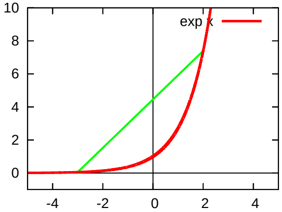

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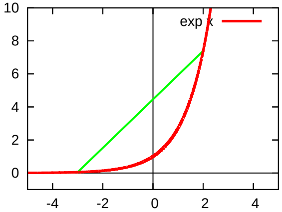

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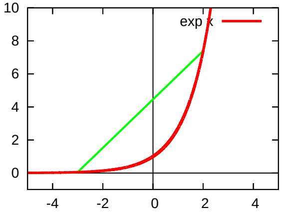

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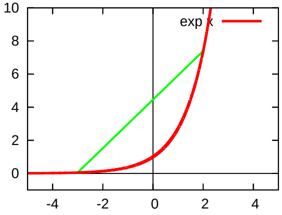

e^x is conve^x!

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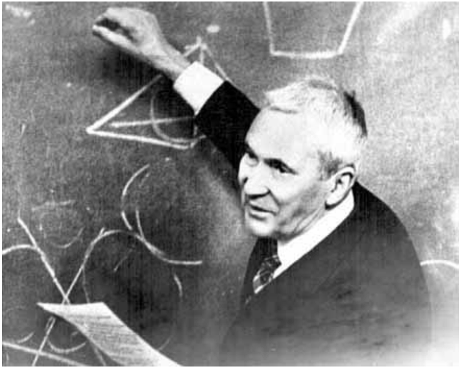
Probability



A.N. Kolmogorov, 1903–1987

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Measure P has to satisfy the **probability axioms**: $P(E) \geq 0$ for all $E \in \mathcal{F}$, $P(\Omega) = 1$, and $P(E_1 \cup E_2 \cup \ldots) = \sum_i P(E_i)$ if (E_i) is a countable sequence of *disjoint* events.

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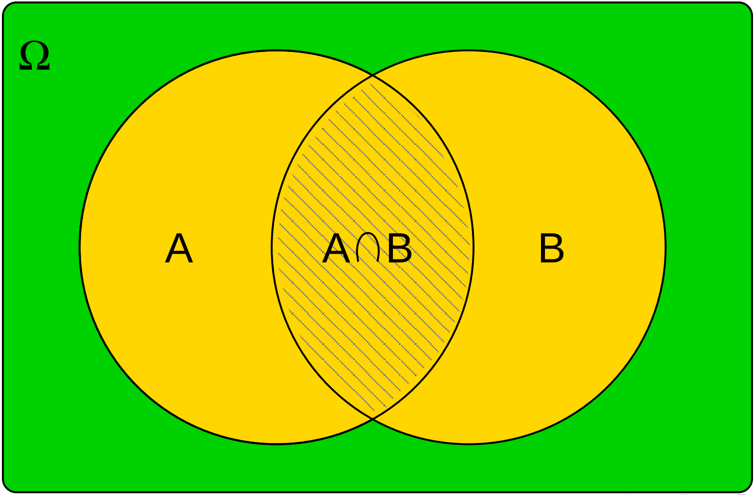
These axioms imply the usual rules of **probability calculus**, e.g., $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, $P(\Omega \backslash E) = 1 - P(E)$, etc.

Venn Diagrams



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1. The **conditional probability** of event B given that event A occurs is defined as

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4. Chain rule:
$$P(\cap_{i=1}^{N} E_i) = \prod_{i=1}^{N} P(E_i \mid \cap_{j=1}^{i-1} E_j)$$
$$= P(E_1) \times P(E_2 \mid E_1) \times P(E_3 \mid E_1 \cap E_2) \times \ldots \times P(E_N \mid E_1 \cap \ldots \cap E_{N-1}) .$$

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In practice, we often forget about the underlying probability space Ω, and just speak of random variable X and its distribution P_X.

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We often omit the subscripts $X, Y, \ldots$ and write $p(x), f(y)$, etc.

Random Variables

Since random variables are functions, we can define more random variables as functions of random variables: if f is a function, and X and Y are r.v.'s, then $f(X) : \Omega \to \mathbb{R}$ is a r.v., $X + Y$ is a r.v., etc.

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! In particular, a pmf p_X is a function, and hence, $p_X(X)$ is also a random variable. Further, $p_X^2(X), \ln p_X(X)$, etc. are random variables.

Multivariate Distributions

The probabilistic behavior of two or more random variables is described by multivariate distributions.

The **joint distribution** of r.v.'s X and Y is

$$\begin{aligned} P_{X,Y}(A,B) &= \Pr[X \in A \wedge Y \in B] \\ &= P(\{\omega : X(\omega) \in A, Y(\omega) \in B\}) . \end{aligned}$$

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For each multivariate distribution $P_{X,Y}$, there are unique **marginal distributions** P_X and P_Y such that

$$P_X(A) = P_{X,Y}(A,\mathbb{R}), \qquad P_Y(B) = P_{X,Y}(\mathbb{R},B) \ ,$$

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For each multivariate distribution $P_{X,Y}$, there are unique **marginal distributions** P_X and P_Y such that

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$$\text{pmf: } p_Y(y) = \sum_{x \in \mathcal{X}} p_{X,Y}(x,y) \quad \text{pdf: } f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)\, dx \; .$$

Multivariate Distributions

The **conditional distribution** is defined similar to *conditional probability*:

$$P_{Y|X}(B \mid A) = \frac{P_{X,Y}(A, B)}{P_X(A)} \quad \text{for } A \text{ such that } P_X(A) > 0.$$

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Independence

Variable X is said to be **independent** of variable Y ($X \perp\!\!\!\perp Y$) iff

$$P_{X,Y}(A, B) = P_X(A) \times P_Y(B) \quad \text{for all } A, B \subseteq \mathbb{R}.$$

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$$P_{Y|X}(B \mid A) = P_Y(B) \quad \text{for all } A \text{ such that } P(A) > 0.$$

In words, knowledge about one variable tells nothing about the other. Note that independence is symmetric, $X \perp\!\!\!\perp Y \Leftrightarrow Y \perp\!\!\!\perp X$.

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The **expectation** (or expected value, or mean) of a discrete random variable is given by

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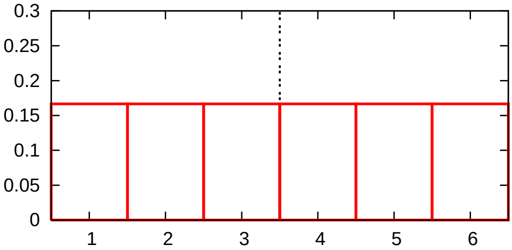
$$E[kX] = kE[X] \qquad E[X+Y] = E[X] + E[Y]$$

$$E[XY] = E[X]E[Y] \quad \text{if } X \perp\!\!\!\perp Y$$

Law of Large Numbers

Let $X_1, X_2, \ldots$ be a sequence of independent outcomes of a die, so that $p_{X_i}(x) = 1/6$ for all $i \in \mathbb{N}, x \in \{1, 2, 3, 4, 5, 6\}$.



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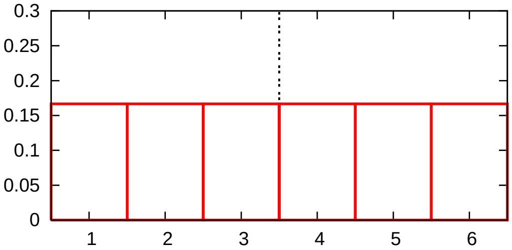

$$E[X_i] = \sum_{x=1}^{6} \frac{1}{6} x = \frac{21}{6} = 3.5 \quad \text{for all } i \in \mathbb{N}.$$

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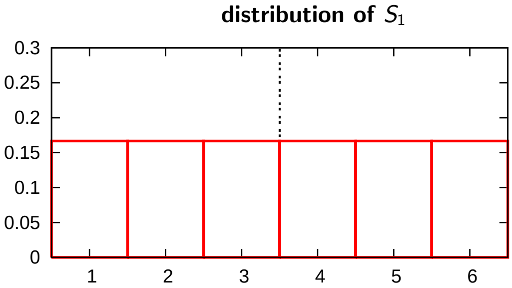

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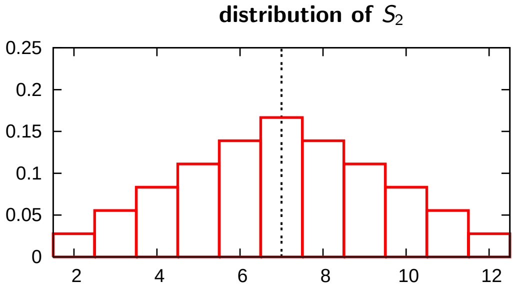

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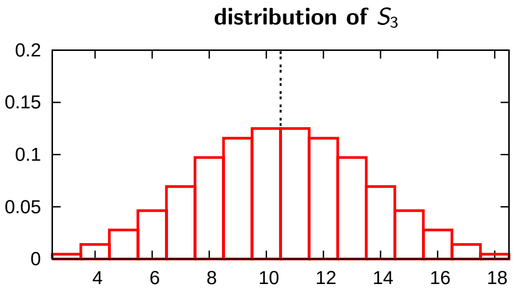

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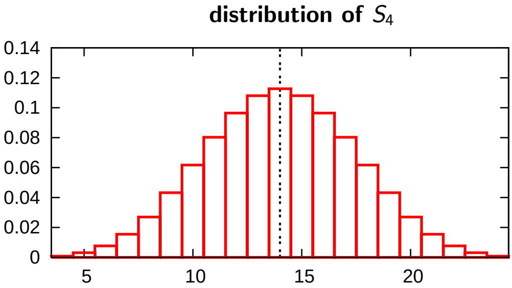

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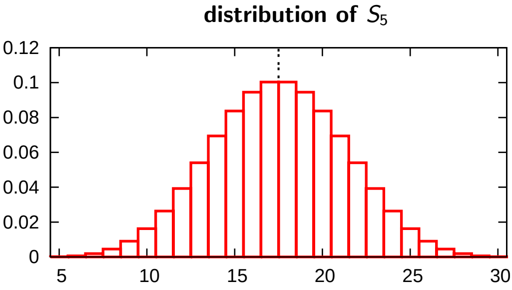

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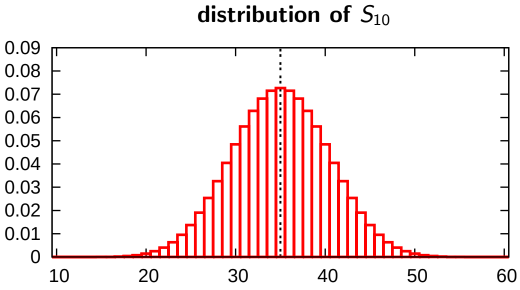

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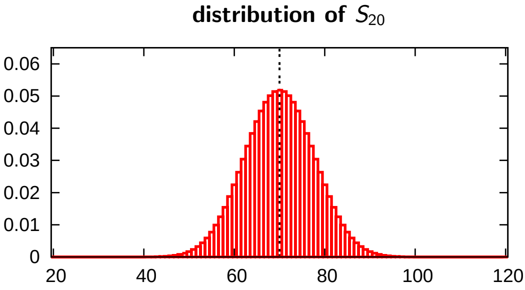

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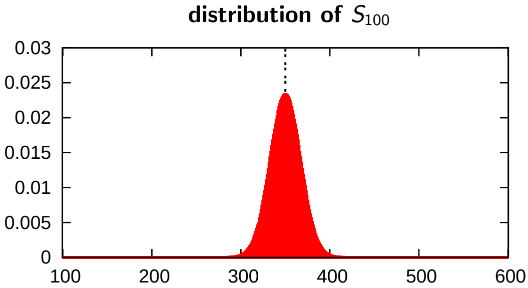

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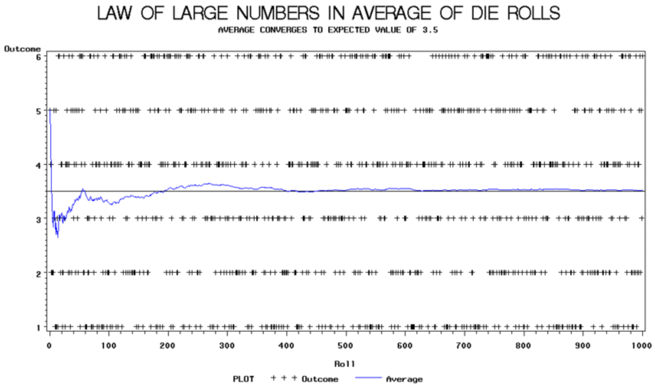

Source: Wikipedia

Law of Large Numbers

Weak Law of Large Numbers

For a sequence of independent and identically distributed (i.i.d.) random variables with finite mean μ, the average $\frac{1}{n}S_n$ converges in probability to μ:

$$\lim_{n\to\infty} \Pr\left[\left|\frac{S_n}{n} - \mu\right| < \epsilon\right] = 1 \quad \text{for all } \epsilon > 0.$$

We will use the LLN to prove a result known as the Asymptotic Equipartition Property (AEP), which is a central result in information theory (see next lecture).

Jensen's inequality

J.L.W.V. Jensen, 1859–1925

Inequalities: Jensen

Jensen's inequality

If f is a convex function and X is a random variable, then

$$E[f(X)] \geq f(E[X]) \ .$$

Moreover, if f is strictly convex, the inequality holds as an equality if and only if $X = E[X]$ with probability 1.

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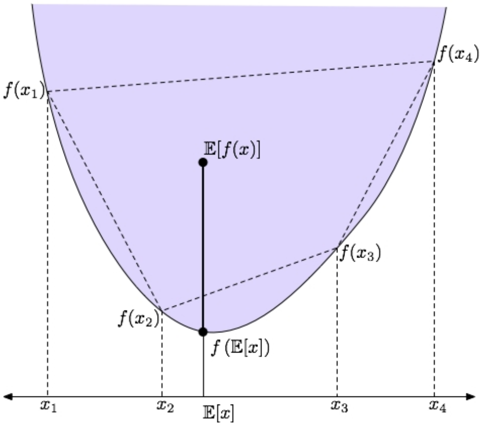

Source: *Inductio Ex Machina*, `mark.reid.name/iem/`

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We give a proof for the first part of the theorem in the special case where X has a finite domain.

For two mass points, we have $p(x_2) = 1 - p(x_1)$, and the claim holds by definition of convexity:

$$p(x_1)\, f(x_1) + p(x_2)\, f(x_2) \geq f(p(x_1)\, x_1 + p(x_2)\, x_2) \ .$$

Inequalities: Jensen

Induction: Assume that $(*)$ the theorem holds for $N-1$ mass points.

$$\sum_{i=1}^{N} p(x_i)\, f(x_i) = p(x_N)\, f(x_N) + (1 - p(x_N)) \sum_{i=1}^{N-1} p'(x_i)\, f(x_i)$$

$$\geq p(x_N)\, f(x_N) + (1 - p(x_N))\, f\left(\sum_{i=1}^{N-1} p'(x_i)\, x_i\right) \quad (*)$$

$$\geq f\left(p(x_N)\, x_N + (1 - p(x_N)) \sum_{i=1}^{N-1} p'(x_i)\, x_i\right) \quad \text{(convexity)}$$

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where $p'(x_i) = \dfrac{p(x_i)}{1 - p(x_N)}$. □

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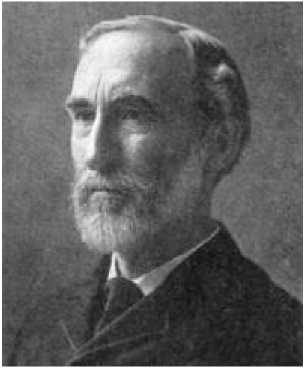
Gibbs' inequality



W. Gibbs, 1839–1903

Inqualities: Gibbs

Gibbs' inequality

For any two discrete probability distributions p and q, we have

$$\sum_{x \in \mathcal{X}} p(x) \log_2 p(x) \geq \sum_{x \in \mathcal{X}} p(x) \log_2 q(x)$$

with equality if and only if $p(x) = q(x)$ for all $x \in \mathcal{X}$.

Proof. Since $\log_2 x = \frac{1}{\ln 2} \ln x$, dividing both sides by $\ln 2$ changes $\log_2$ to $\ln$.

Inqualities: Gibbs

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For any two discrete probability distributions p and q, we have

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&= \sum_{x \in \mathcal{X}} p(x) \ln \frac{q(x)}{p(x)} \qquad \boxed{\ln x - \ln y = \ln \frac{x}{y}} \\
&\leq \sum_{x \in \mathcal{X}} p(x) \left(\frac{q(x)}{p(x)} - 1\right) \qquad \boxed{\ln x \leq x - 1} \\
&= \sum_{x \in \mathcal{X}} q(x) - \sum_{x \in \mathcal{X}} p(x) = 1 - 1 = 0 \ . \qquad \square
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