Information-Theoretic Modeling

Lecture 5: Source Coding: Practice

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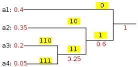
Fall 2009

1. Codes
   - Decodable Codes
   - Prefix Codes
   - Kraft-McMillan Theorem



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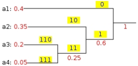
   - Prefix Codes
   - Kraft-McMillan Theorem

2. Optimal Codes
   - Entropy Lower Bound
   - Shannon-Fano Coding



Extension Code

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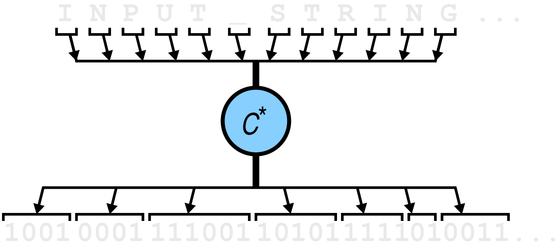

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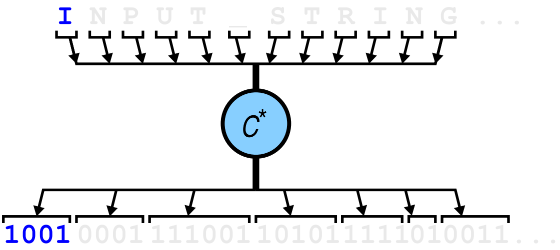

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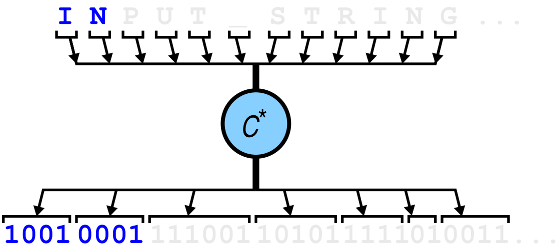

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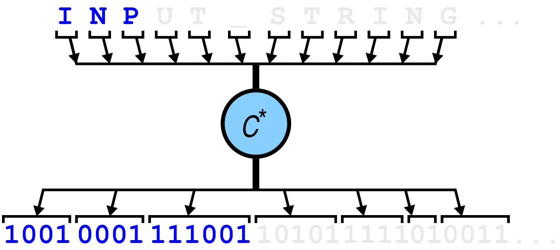

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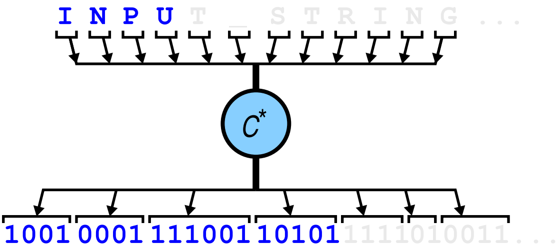

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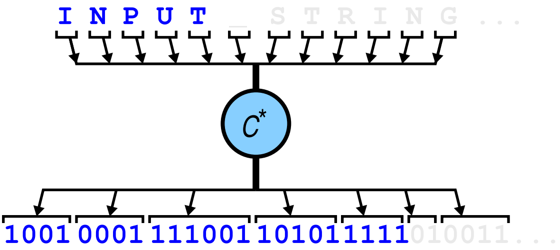

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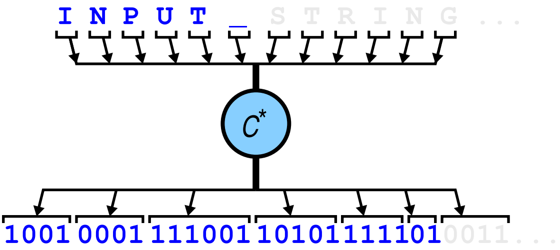

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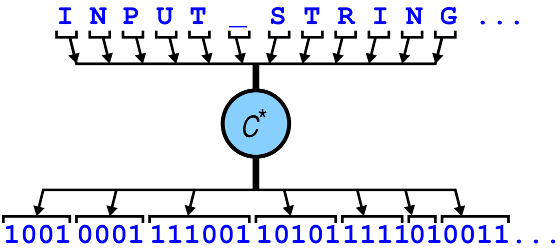

Decodable Codes

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Code C is (uniquely) **decodable** iff its extension C^* is a one-to-one mapping, i.e., iff

$$(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) \;\Rightarrow\; C^*(x_1, \ldots, x_n) \neq C^*(y_1, \ldots, y_n) \;.$$

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- ✗ A code with codewords $\{0, 1, 10, 11\}$ is *not* uniquely decodable: What does 10 mean?
- √ A code with codewords $\{00, 01, 10, 11\}$ *is* uniquely decodable: Each pair of bits can be decoded individually.
- √ A code with codewords $\{0, 01, 011, 0111\}$ is also uniquely decodable: What does 0011 mean?

Prefix Codes

An important subset of decodable codes is the set of **prefix(-free) codes**.

Prefix Code

A code $C : \mathcal{X} \to \{0, 1\}^*$ is called a **prefix code** iff no codeword is a prefix of another.

It is easily seen that all prefix codes are uniquely decodable: each symbol can be decoded as soon as its codeword is read. Therefore, prefix codes are also called *instantaneous* codes.

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- $\surd$ A code with codewords $\{0, 10, 110, 111\}$ *is* prefix-free.

Kraft Inequality

The codeword lengths of a prefix codes satisfy the following important property.

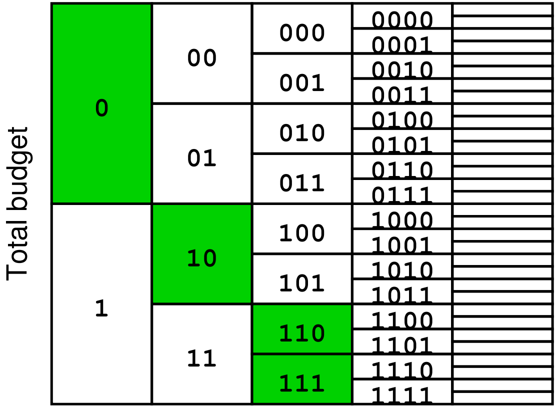
Kraft Inequality

The codeword lengths $\ell_1, \ldots, \ell_m$ of any (binary) prefix code satisfy

$$\sum_{i=1}^{m} 2^{-\ell_i} \leq 1 \ .$$

Conversely, given a set of codeword lengths that satisfy this inequality, there is a prefix code with these codeword lengths.

Kraft Inequality



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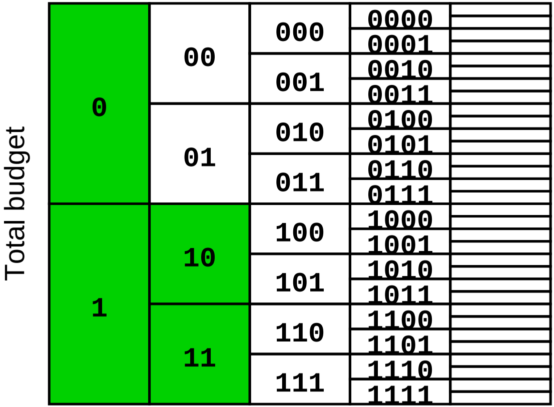
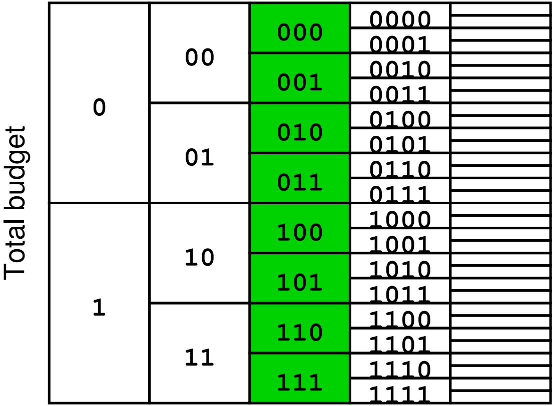


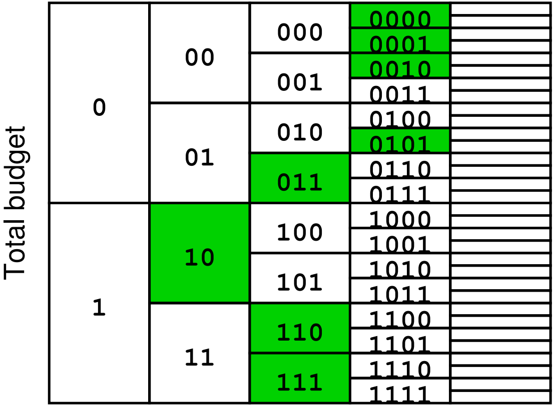
✗ Kraft inequality violated. $\Rightarrow$ Not decodable.

Kraft Inequality



√ Fixed-length code

Kraft Inequality



√ Decodable & prefix-free

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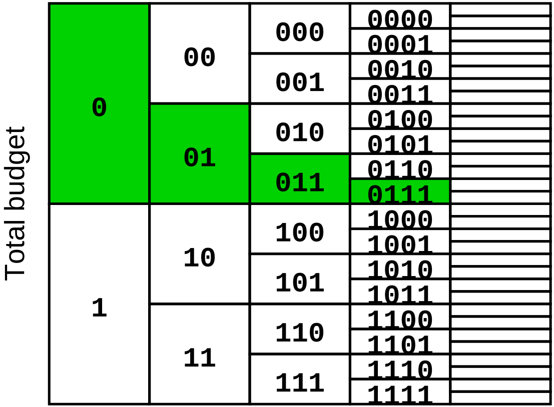

Kraft? Decodable? Prefix-free?

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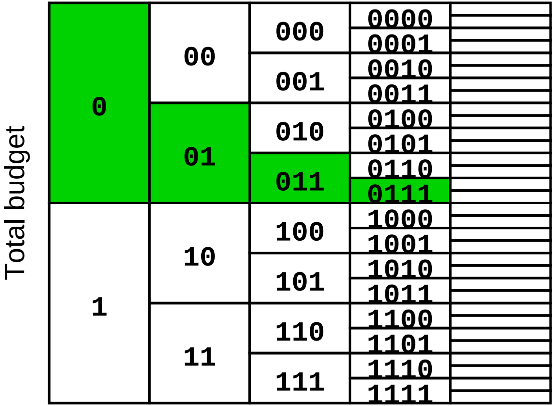

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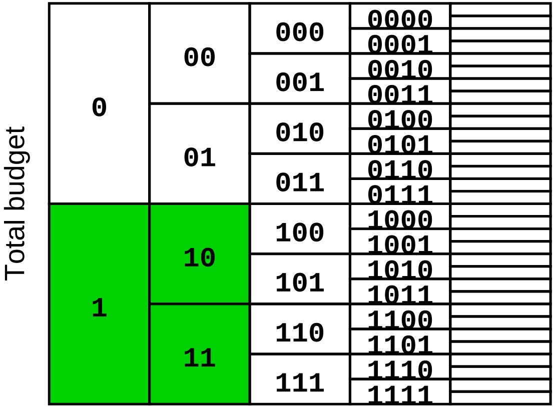

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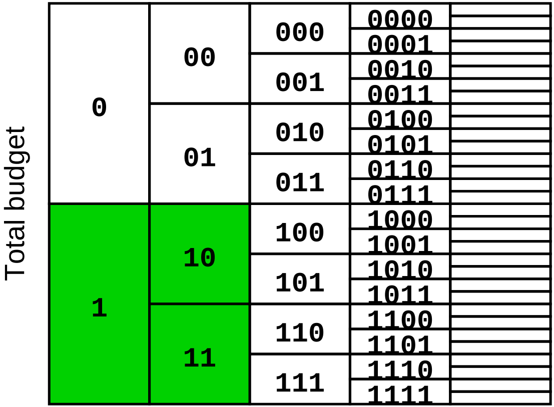

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Question: What if the inequality is satisfied strictly, i.e., the sum of the terms in the sum equals *less* than one:

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Then it is possible to make the codewords shorter and still have a decodable (prefix) code.

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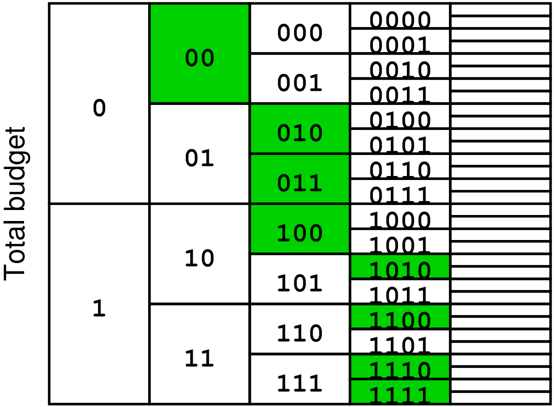

Not all of budget used. $\Rightarrow$ Some codewords can be made shorter.

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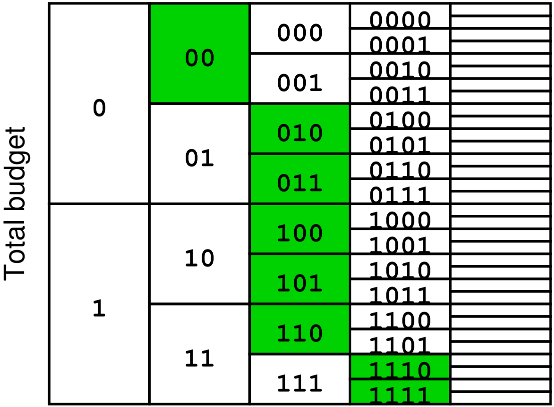

"Kraft tight" / complete code.

Kraft–McMillan Theorem

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The codeword lengths $\ell_1, \ldots, \ell_m$ of any **uniquely decodable** (binary) code satisfy

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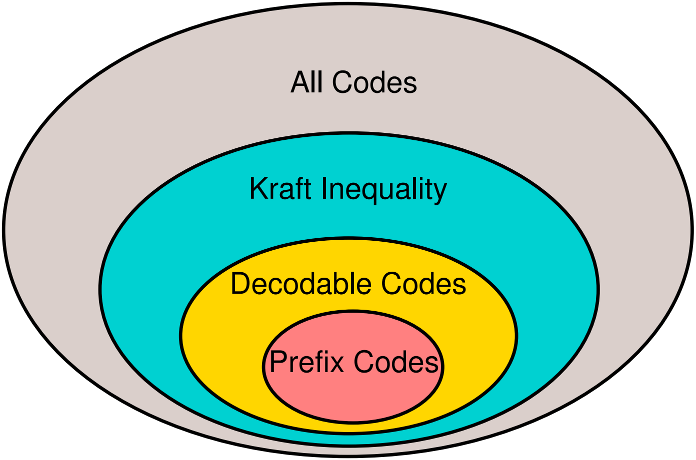
Kraft-McMillan Theorem & Codes



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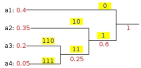

Codelengths and Probabilities

Let $\ell_1, \ldots, \ell_m$ be the codeword lengths of a uniquely decodable code $C : \mathcal{X} \to \{0,1\}^*$. By the Kraft-McMillan theorem we have

$$c = \sum_{i=1}^{m} 2^{-\ell_i} \leq 1 \enspace .$$

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1. Non-negative: $p(x) \geq 0$ for all $x \in \mathcal{X}$.
2. Sums to one: $\sum_{x \in \mathcal{X}} p(x) = \sum_{i=1}^{m} \frac{1}{c} 2^{-\ell_i} = \frac{c}{c} = 1 \ .$

Codelengths and Probabilities

Assuming that the code is "Kraft tight", $c = 1$, then under the pmf p corresponding to the codeword lengths $\ell_1, \ldots, \ell_m$, the expected codeword length is

$$E[\ell(X)] = \sum_{i=1}^{m} 2^{-\ell_i} \, \ell_i$$

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This is the best we can hope for:

The expected codelength of any uniquely decodable code is at least the entropy:

$$E[\ell(X)] \geq H(X) \ .$$

Entropy Lower Bound

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Proof.

$$E[\ell(X)] - H(X) = \sum_{x \in \mathcal{X}} p(x)\,\ell(x) - \sum_{x \in \mathcal{X}} p(x)\,\log_2 \frac{1}{p(x)}$$

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&= \sum_{x \in \mathcal{X}} p(x) \left[\log_2 \frac{p(x)}{q(x)} + \log_2 \frac{1}{c} \right] \qquad \boxed{q(x) = \frac{2^{-\ell(x)}}{c}}
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&= \sum_{x \in \mathcal{X}} p(x) \left[\log_2 \frac{p(x)}{q(x)} + \log_2 \frac{1}{c} \right] \qquad \boxed{q(x) = \frac{2^{-\ell(x)}}{c}} \\
&= D(p \,\|\, q) + \log_2 \frac{1}{c} \geq 0 \ .
\end{aligned}$$

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2. $E[\ell(X)] \geq H(X)$.

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So what have we learned? For decodable symbols codes:

1. $E[\ell(X)] - H(X) = D(p \parallel q) + \log_2 \frac{1}{c}$, where $q(x) = \dfrac{2^{-\ell(x)}}{c}$.
2. $E[\ell(X)] \geq H(X)$.
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Entropy Lower Bound

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$$E[\ell(X_1, \ldots, X_n)] = E\left[\sum_{i=1}^{n} \ell(X_i)\right]$$

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! By Shannon's Noiseless Channel Coding Theorem, this is optimal among all codes, **not only symbol codes.**

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! By Shannon's Noiseless Channel Coding Theorem, this is optimal among all codes, **not only symbol codes.** Fine print: only if X_i i.i.d.!

Codelengths and Probabilities

The only problem with the $\ell(x) = \log_2 \frac{1}{p(x)}$ codeword choice is the requirement that codeword lengths must be **integers** (try to think about a codeword with length 0.123, for instance), while the so obtained ℓ is not in general an integer.

Codelengths and Probabilities

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The simplest solution is to round upwards:

Shannon's Code

Given a pmf, the **Shannon code** has the codeword lengths

$$\ell(x) = \left\lceil \log_2 \frac{1}{p(x)} \right\rceil \quad \text{for all } x \in \mathcal{X}.$$

Alice in Wonderland

Shannon's code: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$
a	0.0644	3.9	4
b	0.0108	6.5	7
c	0.0178	5.8	6
d	0.0359	4.7	5
e	0.0991	3.3	4
f	0.0147	6.0	7
g	0.0184	5.7	6
h	0.0535	4.2	5
i	0.0551	4.1	5
j	0.0011	9.8	10
k	0.0083	6.8	7
l	0.0343	4.8	5
	$\vdots$		
y	0.0165	5.9	6
z	0.0005	10.7	11
	0.2111	2.2	3

$$H(X) = 4.03$$

Shannon's code: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$
a	0.0644	3.9	4
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h	0.0535	4.2	5
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j	0.0011	9.8	10
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Shannon (1948):

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$H(X) = 4.03$

Shannon (1948):

1. Sort by probability.

Shannon's code: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$
	0.2111	2.2	3
e	0.0991	3.3	4
t	0.0781	3.6	4
a	0.0644	3.9	4
o	0.0598	4.0	5
i	0.0551	4.1	5
h	0.0535	4.2	5
n	0.0516	4.2	5
s	0.0475	4.3	5
r	0.0401	4.6	5
d	0.0359	4.7	5
l	0.0343	4.8	5
	$\vdots$		
x	0.0011	9.8	10
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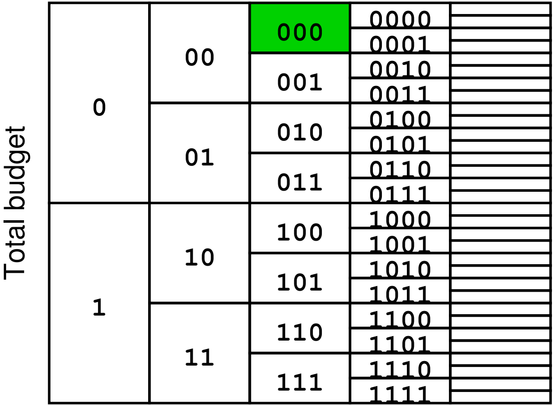
r	0.0401	4.6	5
d	0.0359	4.7	5
l	0.0343	4.8	5
	⋮		
x	0.0011	9.8	10
j	0.0011	9.8	10
z	0.0005	10.7	11

$H(X) = 4.03$

Shannon (1948):

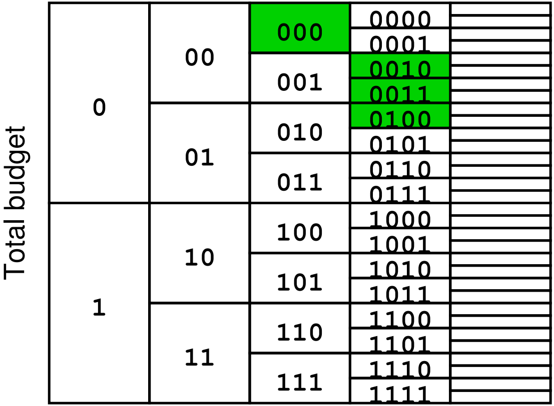
1. Sort by probability.
2. Choose codewords in order, avoiding prefixes. ("Kraft table"!)

Shannon's code: Example



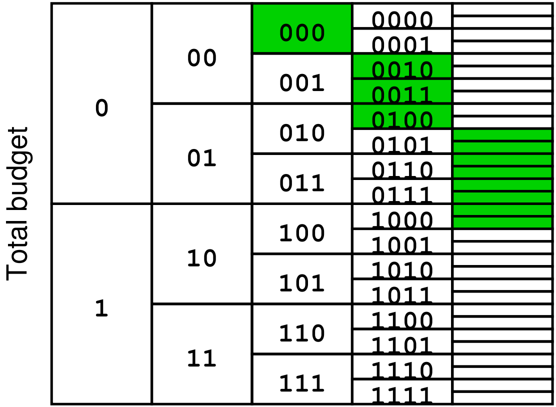
Codeword lengths $(\color{red}{3}, 4, 4, 4, 5, 5, 5, 5, \ldots, 10, 10, 11)$

Shannon's code: Example



Codeword lengths $(3, 4, 4, 4, 5, 5, 5, 5, \ldots, 10, 10, 11)$

Shannon's code: Example



Codeword lengths $(3, 4, 4, 4, 5, 5, 5, 5, \ldots, 10, 10, 11)$

Shannon's code: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	3	000
e	0.0991	3.3	4	0010
t	0.0781	3.6	4	0011
a	0.0644	3.9	4	0100
o	0.0598	4.0	5	01010
i	0.0551	4.1	5	01011
h	0.0535	4.2	5	01100
n	0.0516	4.2	5	01101
s	0.0475	4.3	5	01110
r	0.0401	4.6	5	01111
d	0.0359	4.7	5	10000
l	0.0343	4.8	5	10001
	⋮			
x	0.0011	9.8	10	1010111101
j	0.0011	9.8	10	1010111110
z	0.0005	10.7	11	10101111110

Shannon's code: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	3	000
e	0.0991	3.3	4	0010
t	0.0781	3.6	4	0011
a	0.0644	3.9	4	0100
o	0.0598	4.0	5	01010
i	0.0551	4.1	5	01011
h	0.0535	4.2	5	01100
n	0.0516	4.2	5	01101
s	0.0475	4.3	5	01110
r	0.0401	4.6	5	01111
d	0.0359	4.7	5	10000
l	0.0343	4.8	5	10001
	$\vdots$			
x	0.0011	9.8	10	1010111101
j	0.0011	9.8	10	1010111110
z	0.0005	10.7	11	10101111110

$$H(X) = 4.03$$
$$E[\ell(X)] = 4.60$$
$$E[\ell(X)] - H(X) = 0.57$$

Shannon's code

The expected codeword length of Shannon's code is

$$E\left[\ell(X)\right] = E\left[\left\lceil \log_2 \frac{1}{p(X)} \right\rceil\right]$$
$$\leq E\left[\log_2 \frac{1}{p(X)} + 1\right] = H(X) + 1 \quad .$$

Shannon's code

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$$E\left[\ell(X)\right] = E\left[\left\lceil \log_2 \frac{1}{p(X)} \right\rceil\right]$$
$$\leq E\left[\log_2 \frac{1}{p(X)} + 1\right] = H(X) + 1 \ .$$

In the Alice example we had

$$E[\ell(X)] - H(X) = 4.60 - 4.03 = 0.57 \leq 1 \ .$$

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$
a	0.0644	3.9
b	0.0108	6.5
c	0.0178	5.8
d	0.0359	4.7
e	0.0991	3.3
f	0.0147	6.0
g	0.0184	5.7
h	0.0535	4.2
i	0.0551	4.1
j	0.0011	9.8
k	0.0083	6.8
l	0.0343	4.8
	$\vdots$	
y	0.0165	5.9
z	0.0005	10.7
	0.2111	2.2

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X	$p(X)$	$\log_2 \frac{1}{p(X)}$
a	0.0644	3.9
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i	0.0551	4.1
j	0.0011	9.8
k	0.0083	6.8
l	0.0343	4.8
	$\vdots$	
y	0.0165	5.9
z	0.0005	10.7
	0.2111	2.2

(Shannon-)Fano code:

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$
a	0.0644	3.9
b	0.0108	6.5
c	0.0178	5.8
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e	0.0991	3.3
f	0.0147	6.0
g	0.0184	5.7
h	0.0535	4.2
i	0.0551	4.1
j	0.0011	9.8
k	0.0083	6.8
l	0.0343	4.8
	⋮	
y	0.0165	5.9
z	0.0005	10.7
	0.2111	2.2

(Shannon-)Fano code:

1. Sort by probability.

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$
	0.2111	2.2
e	0.0991	3.3
t	0.0781	3.6
a	0.0644	3.9
o	0.0598	4.0
i	0.0551	4.1
h	0.0535	4.2
n	0.0516	4.2
s	0.0475	4.3
r	0.0401	4.6
d	0.0359	4.7
l	0.0343	4.8
	$\vdots$	
x	0.0011	9.8
j	0.0011	9.8
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a	0.0644	3.9
o	0.0598	4.0
i	0.0551	4.1
h	0.0535	4.2
n	0.0516	4.2
s	0.0475	4.3
r	0.0401	4.6
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l	0.0343	4.8
	⋮	
x	0.0011	9.8
j	0.0011	9.8
z	0.0005	10.7

(Shannon-)Fano code:

1. Sort by probability.
2. Divide in two equally probable parts (as equal as possible)

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$
	0.2111	2.2
e	0.0991	3.3
t	0.0781	3.6
a	0.0644	3.9
o	0.0598	4.0
i	0.0551	4.1
h	0.0535	4.2
n	0.0516	4.2
s	0.0475	4.3
r	0.0401	4.6
d	0.0359	4.7
l	0.0343	4.8
	$\vdots$	
x	0.0011	9.8
j	0.0011	9.8
z	0.0005	10.7

(Shannon-)Fano code:

1. Sort by probability.
2. Divide in two equally probable parts (as equal as possible)
3. Add '0' to the codewords in the first part, '1' to the others.

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$
	0.2111	2.2
e	0.0991	3.3
t	0.0781	3.6
a	0.0644	3.9
o	0.0598	4.0
i	0.0551	4.1
h	0.0535	4.2
n	0.0516	4.2
s	0.0475	4.3
r	0.0401	4.6
d	0.0359	4.7
l	0.0343	4.8
	⋮	
x	0.0011	9.8
j	0.0011	9.8
z	0.0005	10.7

(Shannon-)Fano code:

1. Sort by probability.
2. Divide in two equally probable parts (as equal as possible)
3. Add '0' to the codewords in the first part, '1' to the others.
4. Repeat recursively for both parts.

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	0
e	0.0991	3.3	4	0
t	0.0781	3.6	4	0
a	0.0644	3.9	4	0
o	0.0598	4.0	4	0
i	0.0551	4.1	4	1
h	0.0535	4.2	4	1
n	0.0516	4.2	4	1
s	0.0475	4.3	5	1
r	0.0401	4.6	5	1
d	0.0359	4.7	5	1
l	0.0343	4.8	5	1
	$\vdots$			
x	0.0011	9.8	10	1
j	0.0011	9.8	10	1
z	0.0005	10.7	10	1

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	00
e	0.0991	3.3	4	01
t	0.0781	3.6	4	01
a	0.0644	3.9	4	01
o	0.0598	4.0	4	01
i	0.0551	4.1	4	10
h	0.0535	4.2	4	10
n	0.0516	4.2	4	10
s	0.0475	4.3	5	10
r	0.0401	4.6	5	10
d	0.0359	4.7	5	11
l	0.0343	4.8	5	11
	⋮			
x	0.0011	9.8	10	11
j	0.0011	9.8	10	11
z	0.0005	10.7	10	11

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	00
e	0.0991	3.3	4	010
t	0.0781	3.6	4	010
a	0.0644	3.9	4	011
o	0.0598	4.0	4	011
i	0.0551	4.1	4	100
h	0.0535	4.2	4	100
n	0.0516	4.2	4	101
s	0.0475	4.3	5	101
r	0.0401	4.6	5	101
d	0.0359	4.7	5	110
l	0.0343	4.8	5	110
	⋮			
x	0.0011	9.8	10	111
j	0.0011	9.8	10	111
z	0.0005	10.7	10	111

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	00
e	0.0991	3.3	4	0100
t	0.0781	3.6	4	0101
a	0.0644	3.9	4	0110
o	0.0598	4.0	4	0111
i	0.0551	4.1	4	1000
h	0.0535	4.2	4	1001
n	0.0516	4.2	4	1010
s	0.0475	4.3	5	1011
r	0.0401	4.6	5	1011
d	0.0359	4.7	5	1100
l	0.0343	4.8	5	1100
	$\vdots$			
x	0.0011	9.8	10	1111
j	0.0011	9.8	10	1111
z	0.0005	10.7	10	1111

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	00
e	0.0991	3.3	4	0100
t	0.0781	3.6	4	0101
a	0.0644	3.9	4	0110
o	0.0598	4.0	4	0111
i	0.0551	4.1	4	1000
h	0.0535	4.2	4	1001
n	0.0516	4.2	4	1010
s	0.0475	4.3	5	10110
r	0.0401	4.6	5	10111
d	0.0359	4.7	5	11000
l	0.0343	4.8	5	11001
	⋮			
x	0.0011	9.8	10	11111
j	0.0011	9.8	10	11111
z	0.0005	10.7	10	11111

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	00
e	0.0991	3.3	4	0100
t	0.0781	3.6	4	0101
a	0.0644	3.9	4	0110
o	0.0598	4.0	4	0111
i	0.0551	4.1	4	1000
h	0.0535	4.2	4	1001
n	0.0516	4.2	4	1010
s	0.0475	4.3	5	10110
r	0.0401	4.6	5	10111
d	0.0359	4.7	5	11000
l	0.0343	4.8	5	11001
	$\vdots$			
x	0.0011	9.8	10	111111
j	0.0011	9.8	10	111111
z	0.0005	10.7	10	111111

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	00
e	0.0991	3.3	4	0100
t	0.0781	3.6	4	0101
a	0.0644	3.9	4	0110
o	0.0598	4.0	4	0111
i	0.0551	4.1	4	1000
h	0.0535	4.2	4	1001
n	0.0516	4.2	4	1010
s	0.0475	4.3	5	10110
r	0.0401	4.6	5	10111
d	0.0359	4.7	5	11000
l	0.0343	4.8	5	11001
	⋮			
x	0.0011	9.8	10	1111111101
j	0.0011	9.8	10	1111111110
z	0.0005	10.7	10	1111111111

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	00
e	0.0991	3.3	4	0100
t	0.0781	3.6	4	0101
a	0.0644	3.9	4	0110
o	0.0598	4.0	4	0111
i	0.0551	4.1	4	1000
h	0.0535	4.2	4	1001
n	0.0516	4.2	4	1010
s	0.0475	4.3	5	10110
r	0.0401	4.6	5	10111
d	0.0359	4.7	5	11000
l	0.0343	4.8	5	11001
	$\vdots$			
x	0.0011	9.8	10	1111111101
j	0.0011	9.8	10	1111111110
z	0.0005	10.7	10	1111111111

$$H(X) = 4.03$$
$$E[\ell(X)] = 4.07$$
$$E[\ell(X)] - H(X) = 0.04$$

Shannon-Fano Code

The expected codeword length of the Shannon-Fano code is

$$\begin{aligned} E\left[\ell(X)\right] &\leq E\left[\left\lceil \log_2 \frac{1}{p(X)} \right\rceil\right] \\ &\leq E\left[\log_2 \frac{1}{p(X)} + 1\right] = H(X) + 1 \ . \end{aligned}$$

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Is this optimal? Not necessarily — Huffman!

What's next?

Friday:

- Daniel Schmidt's guest lecture on the **Minimum Message Length (MML) principle**.

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- Daniel Schmidt's guest lecture on the **Minimum Message Length (MML) principle**.

Next week:

- the optimal prefix code: Huffman,
- even better: Rissanen's arithmetic coding.