Information-Theoretic Modeling

Lecture 10: MDL Principle — Part II

Teemu Roos

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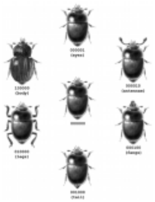
Fall 2009

1. MDL for Gaussian Models
   - Encoding Continuous Data
   - Differential Entropy
   - Linear Regression
   - Subset Selection Problem
   - Wavelet Denoising



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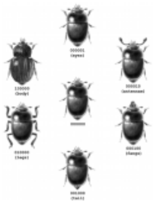
2. MDL for Multinomial Models
   - Universal Codes
   - Fast NML Computation
   - Histogram Density Estimation
   - Clustering



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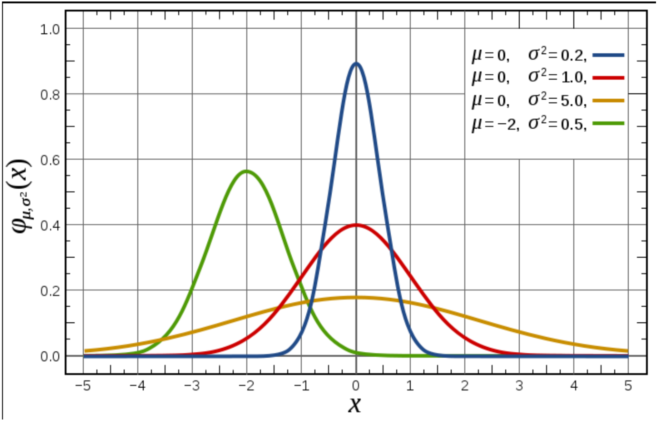

Source: Wikipedia

Gaussian models

Density function:

$$\phi_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\dfrac{(x-\mu)^2}{2\sigma^2}} .$$

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Mean: $\mu = E[X]$, variance $\sigma^2 = E[(X-\mu)^2]$

Maximum likelihood: $\hat{\mu} = \dfrac{1}{n}\displaystyle\sum_{i=1}^n x_i$, $\hat{\sigma}^2 = \dfrac{1}{n}\displaystyle\sum_{i=1}^n (x_i-\hat{\mu})^2$.

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Obviously not possible to encode data with infinite precision. Have to **discretize**: encode x only up to precision δ.

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Let $X \in \mathbb{R}$ be a continuous random variable with probability density $f : \mathbb{R} \to \mathbb{R}^+$.

The differential entropy of X is defined as

$$h(X) = E_{X \sim f}\left[\log_2 \frac{1}{f(X)}\right] = \int f(x) \log_2 \frac{1}{f(x)}\, dx.$$

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Hence, the rate is approximately $H(X^\delta) \approx h(X) - \log_2 \delta$.

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Recall the Gaussian density function:

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The code-length is then

$$\frac{n}{2}\log_2(2\pi\sigma^2) - \frac{1}{(2\ln 2)\sigma^2}\sum_{i=1}^n (x_i-\mu)^2.$$

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Ok, we have our Gaussian code-length formula:

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$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i , \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2 .$$

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The dependent variable, Y, is assumed to be Gaussian:

- the mean μ is given as a linear combination of the regressors:

$$\mu = \beta_1 x_1 + \cdots + \beta_p x_p = \beta' x,$$

- variance is some parameter σ^2.

Linear Regression

For a sample of size n, the matrix notation is convenient:

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \quad X = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{np} \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} \quad \epsilon = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

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Then the model can be written as

$$Y = X\beta + \epsilon,$$

where $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$.

Linear Regression

The maximum likelihood estimators are now

$$\hat{\beta} = (X'X)^{-1}X'Y, \quad \hat{\sigma}^2 = \frac{1}{n}\|Y - X\hat{\beta}\|_2^2 = \frac{\mathrm{RSS}}{n},$$

where RSS is the "residual sum of squares".

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$\Rightarrow$ Exercise 5.3

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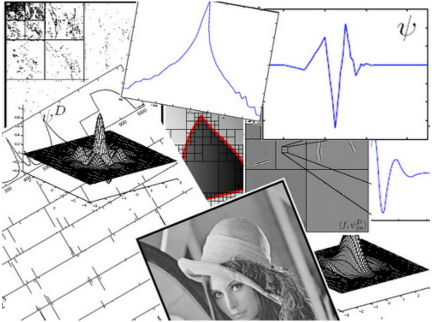

Image by Gabriel Peyré

Wavelet Denoising

IEEE TRANS. SIGNAL PROCESSING, VOL. ?, NO. ?, 2009

MDL Denoising Revisited

Teemu Roos *Member*, Petri Myllymäki, and Jorma Rissanen *Fellow*

***Abstract*— We refine and extend an earlier minimum description length (MDL) denoising criterion for wavelet-based denoising. We start by showing that the denoising problem can be reformulated as a clustering problem, where the goal is to obtain separate clusters for informative and non-informative wavelet coefficients, respectively. This suggests two refinements, adding a code-length for the model index, and extending the model in order to account for subband-dependent coefficient distributions. A third refinement is the derivation of soft thresholding inspired by predictive universal coding with weighted mixtures. We propose a practical method incorporating all three refinements, which is shown to achieve good performance and robustness in denoising both artificial and natural signals.**

***Index Terms*— Minimum description length (MDL) principle, wavelets, denoising.**

(both of which include the Gaussian and d densities as special cases).

A third approach to denoising is based description length (MDL) principle [16]–[2 ent MDL denoising methods have been su [21]–[25]. We focus on what we consider MDL approach, namely that of Rissanen [24 is two-fold: First, as an immediate result extending the earlier MDL denoising meth new practical method with greatly impro and robustness. Secondly, the denoising p to illustrate theoretical issues related to the involving the problem of unbounded paran and the necessity of encoding the model cl

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1. combines two-part, mixture, and NML universal codes,
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3. important lesson: remember to encode model class.

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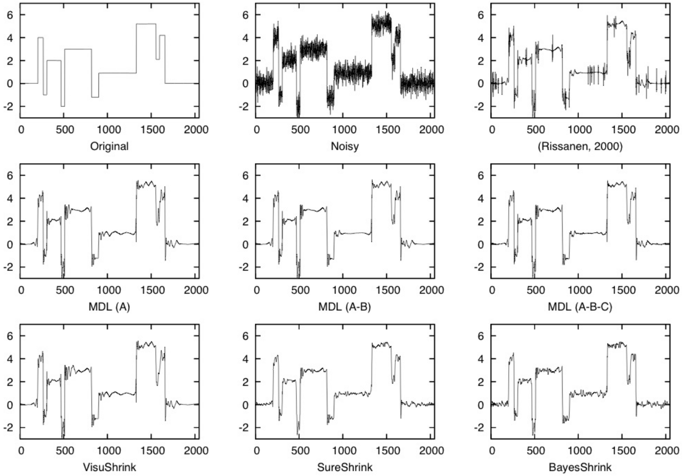

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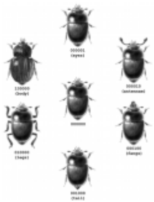

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Two-part, mixture, and NML models readily defined.
⇒ Exercises 5.1 & 5.2

Fast NML for Multinomials

The naïve way to compute the normalizing constant in the NML model

$$\frac{p_{\hat{\theta}}(x^n)}{C_n^m}, \qquad C_n^m = \sum_{y^n \in \mathcal{X}^n} p_{\hat{\theta}}(y^n),$$

takes exponential time ($\Omega(m^n)$).

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The second most naïve way takes "only" polynomial time, $O(n^{m-1})$, but is still intractable unless $m \leq 3$ (or maybe $m \leq 4$).

Fast NML for Multinomials

There is a way — which is not naïve at all! — to do it in linear time, $O(n+m)$, using the following recursion:

$$C_n^m = C_n^{m-1} + \frac{n}{m-2} C_n^{m-2},$$

where C_n^m is the normalizing constant for an m-ary multinomial and sample size n.

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Kontkanen & Myllymäki, "A linear-time algorithm for computing the multinomial stochastic complexity", *Information Processing Letters* **103** (2007), 6, pp. 227–233

Histogram Density Estimation

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Choosing the number *and the positions* of break-points can be done by MDL.

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$\Rightarrow$ Linear time algorithm can be used.

Histogram Density Estimation

MDL Histogram Density Estimation

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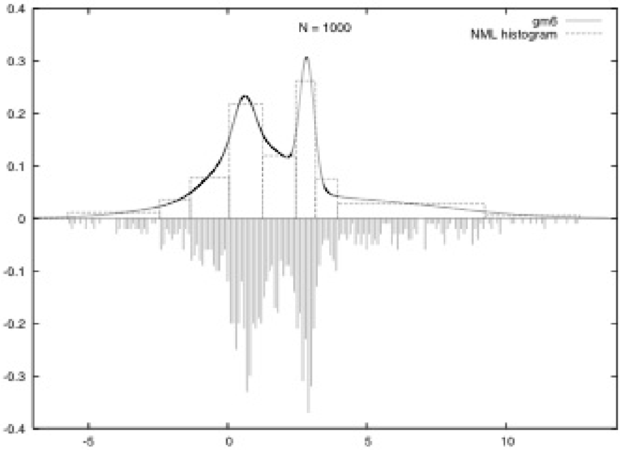
Abstract

We regard histogram density estimation as a model selection problem. Our approach is based on the information-theoretic minimum description length (MDL) principle, which can be applied for tasks such as data clustering, density estimation, image denois-

only on finding the optimal bin count. These *regular* histograms are, however, often problematic. It has been argued (Rissanen, Speed, & Yu, 1992) that regular histograms are only good for describing roughly uniform data. If the data distribution is strongly non-uniform, the bin count must necessarily be high if one wants to capture the details of the high density portion of the data. This in turn means that an unnecessary large amount of bins is wasted in the low density re-

Histogram Density Estimation



Clustering

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Again, the problem is reduced to the multinomial case, and the fast NML algorithm can be applied.

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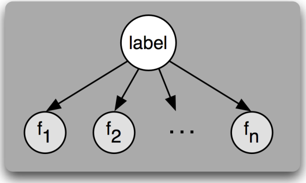
The clustering model can be interpreted as the **naïve Bayes** structure:



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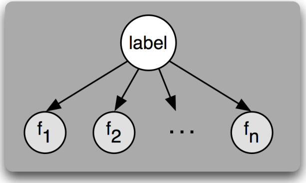

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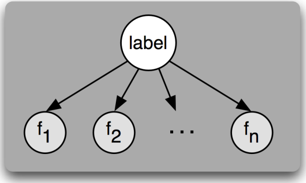

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MDL criterion for learning Bayesian network structures (Lecture 9) again based on *fast NML for multinomials*.

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