Information-Theoretic Modeling

Lecture 3: Source Coding: Theory

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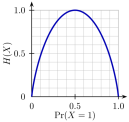
1. Entropy and Information
   - Entropy
   - Information Inequality
   - Data Processing Inequality

2. Data Compression
   - Asymptotic Equipartition Property (AEP)
   - Typical Sets
   - Noiseless Source Coding Theorem



Entropy

Given a discrete random variable X with pmf p_X, we can measure the amount of "surprise" associated with each outcome $x \in \mathcal{X}$ by the quantity

$$I_X(x) = \log_2 \frac{1}{p_X(x)} \quad .$$

The less likely an outcome is, the more surprised we are to observe it. (The point in the log-scale will become clear shortly.)

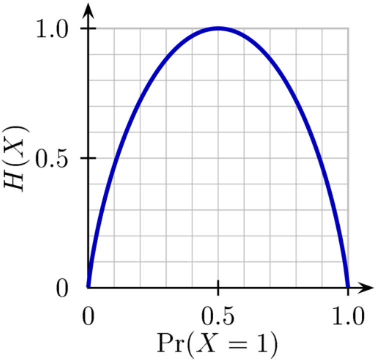
The **entropy** of X measures the *expected* amount of "surprise":

$$H(X) = E[I_X(X)] = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{1}{p_X(x)} \quad .$$

Binary Entropy Function

For binary-valued X, with $p = p_X(1) = 1 - p_X(0)$, we have

$$H(X) = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p} \ .$$



More Entropies

1. the **joint entropy** of two (or more) random variables:

$$H(X,Y) = \sum_{\substack{x\in\mathcal{X}\\ y\in\mathcal{Y}}} p_{X,Y}(x,y)\,\log_2 \frac{1}{p_{X,Y}(x,y)}\ ,$$

2. the **entropy of a conditional distribution**:

$$H(X \mid Y=y) = \sum_{x\in\mathcal{X}} p_{X|Y}(x\mid y)\,\log_2 \frac{1}{p_{X|Y}(x\mid y)}\ ,$$

3. and the **conditional entropy**:

$$\begin{aligned} H(X\mid Y) &= \sum_{y\in\mathcal{Y}} p(y)\, H(X\mid Y=y) \\ &= \sum_{\substack{x\in\mathcal{X}\\ y\in\mathcal{Y}}} p_{X,Y}(x,y)\,\log_2 \frac{1}{p_{X|Y}(x\mid y)}\ . \end{aligned}$$

More Entropies

The joint entropy $H(X, Y)$ measures the uncertainty about the pair (X, Y).

The entropy of the conditional distribution $H(X \mid Y = y)$ measures the uncertainty about X when we know that $Y = y$.

The conditional entropy $H(X \mid Y)$ measures the *expected* uncertainty about X when the value Y is known.

Chain Rule of Entropy

Remember the chain rule of probability:

$$p_{X,Y}(x, y) = p_Y(y) \cdot p_{X|Y}(x \mid y) \ .$$

For the entropy we have:

Chain Rule of Entropy

$$H(X, Y) = H(Y) + H(X \mid Y) \ .$$

Proof.

$$p_{X,Y}(x, y) = p_Y(y) \cdot p_{X|Y}(x \mid y)$$

Next apply $\log(ab) = \log a + \log b$.

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Proof.

$$\log_2 p_{X,Y}(x,y) = \log_2 p_Y(y) + \log_2 p_{X|Y}(x \mid y)$$

Next apply $\log a = -\log(1/a)$.

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Proof.

$$\log_2 \frac{1}{p_{X,Y}(x,y)} = \log_2 \frac{1}{p_Y(y)} + \log_2 \frac{1}{p_{X|Y}(x \mid y)}$$

$$\Leftrightarrow E\left[\log_2 \frac{1}{p_{X,Y}(x,y)}\right] = E\left[\log_2 \frac{1}{p_Y(y)}\right] + E\left[\log_2 \frac{1}{p_{X|Y}(x \mid y)}\right]$$

$$\Leftrightarrow H(X,Y) = H(Y) + H(X \mid Y) \ .$$

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$$H(X,Y) = H(Y) + H(X \mid Y) \ .$$

The rule can be extended to more than two random variables:

$$H(X_1,\ldots,X_n) = \sum_{i=1}^{n} H(X_i \mid H_1,\ldots,H_{i-1}) \ .$$

$$X \perp\!\!\!\perp Y \ \Leftrightarrow \ H(X \mid Y) = H(X) \ \Leftrightarrow \ H(X,Y) = H(X) + H(Y).$$

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Logarithmic scale makes entropy **additive**.

Mutual Information

The **mutual information**

$$I(X\,;\,Y) = H(X) - H(X \mid Y)$$

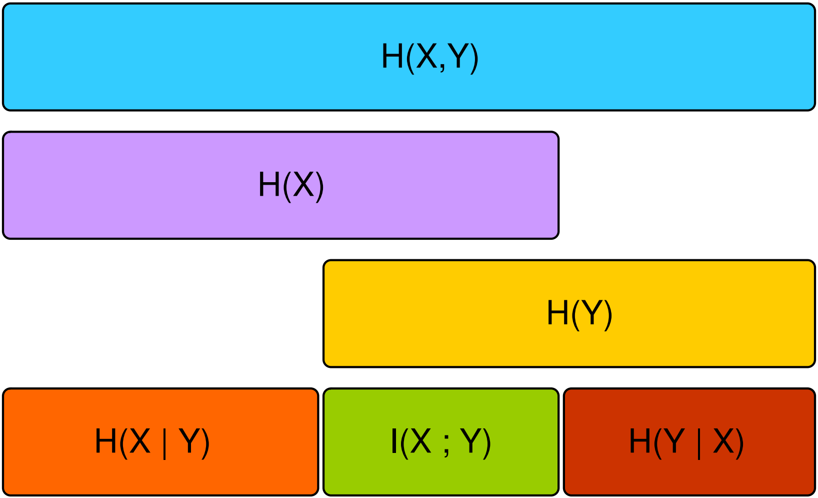
measures the average decrease in uncertainty about X when the value of Y becomes known.

Mutual information is *symmetric* (chain rule):

$$\begin{aligned} I(X\,;\,Y) &= H(X) - H(X \mid Y) = H(X) - (H(X,Y) - H(Y)) H(X) - H(X, \\ &= H(Y) - H(Y \mid X) = I(Y\,;\,X) \ . \end{aligned}$$

On the average, X gives as much information about Y as Y gives about X.

Relationships between Entropies



Information Inequality

Kullback-Leibler Divergence

The *relative entropy* or **Kullback-Leibler divergence** between (discrete) distributions p_X and q_X is defined as

$$D(p_X \parallel q_X) = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{p_X(x)}{q_X(x)} \quad .$$

(We consider $p_X(x) \log_2 \frac{p_X(x)}{q_X(x)} = 0$ whenever $p_X(x) = 0$.)

Information Inequality

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$$D(p_X \parallel q_X) = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{p_X(x)}{q_X(x)} \quad .$$

Information Inquality

For any two (discrete) distributions p_X and q_X, we have

$$D(p_X \parallel q_X) \geq 0$$

with equality iff $p_X(x) = q_X(x)$ for all $x \in \mathcal{X}$.

Proof. Gibbs!

Kullback-Leibler Divergence

The information inequality implies

$$I(X\,;\,Y) \geq 0\ .$$

Proof.

$$\begin{aligned} I(X\,;\,Y) &= H(X) - H(X \mid Y) \\ &= H(X) + H(Y) - H(X, Y) \\ &= \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} p_{X,Y}(x, y) \log_2 \frac{p_{X,Y}(x, y)}{p_X(x)\, p_Y(y)} \\ &= D(p_{X,Y} \parallel p_X p_Y) \geq 0\ . \end{aligned}$$

In addition, $D(p_{X,Y} \parallel p_X p_Y) = 0$ iff $p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$ for all $x \in \mathcal{X}, y \in \mathcal{Y}$. This means that variables X and Y are *independent* iff $I(X\,;\,Y) = 0$.

Properties of Entropy

Properties of entropy:

1. $H(X) \geq 0$

   Proof. $p_X(x) \leq 1 \Rightarrow \log_2 \frac{1}{p_X(x)} \geq 0.$

2. $H(X) \leq \log_2 |\mathcal{X}|$

   Proof. Let $u_X(x) = \frac{1}{|\mathcal{X}|}$ be the uniform distribution over $\mathcal{X}$.

$$0 \leq D(p_X \parallel u_X) = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{p_X(x)}{u_X(x)} = \log_2 |\mathcal{X}| - H(X) \quad .$$

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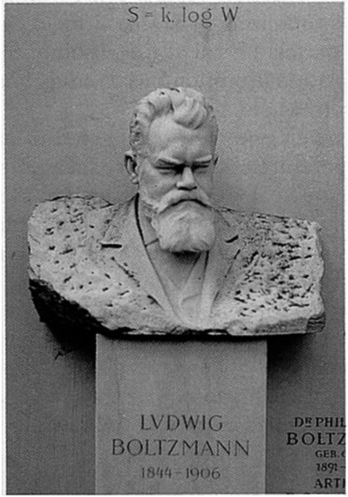
   A **combinatorial** approach to the definition of information (Boltzmann, 1896; Hartley, 1928; Kolmogorov, 1965):

$$S = k \ln W \quad .$$

Ludvig Boltzmann (1844–1906)



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   $$S = k \ln W \quad .$$

3. $H(X \mid Y) \leq H(X)$

   Proof.

   $$0 \leq I(X \; ; \; Y) = H(X) - H(X \mid Y) \quad .$$

Properties of Entropy

Properties of entropy:

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   A **combinatorial** approach to the definition of information (Boltzmann, 1896; Hartley, 1928; Kolmogorov, 1965):

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3. $H(X \mid Y) \leq H(X)$

   On the average, knowing another r.v. can only reduce uncertainty about X. However, note that $H(X \mid Y = y)$ may be greater than $H(X)$ for some y — "contradicting evidence".

Chain Rule of Mutual Information

The **conditional mutual information** of variables X and Y given Z is defined as

$$I(X\,;\,Y \mid Z) = H(X \mid Z) - H(X \mid Y, Z)\ .$$

Chain Rule of Mutual Information

For random variables X and $Y_1, \ldots, Y_n$ we have

$$I(X\,;\,Y_1, \ldots, Y_n) = \sum_{i=1}^{n} I(X\,;\,Y_i \mid Y_1, \ldots, Y_{i-1})\ .$$

Independence among $Y_1, \ldots, Y_n$ implies

$$I(X\,;\,Y_1, \ldots, Y_n) = \sum_{i=1}^{n} I(X\,;\,Y_i)\ .$$

Data Processing Inequality

Let X, Y, Z be (discrete) random variables. If Z is *conditionally independent of X given Y*, i.e., if we have

$$p_{Z|X,Y}(z \mid x, y) = p_{Z|Y}(z \mid y) \quad \text{for all } x, y, z,$$

then X, Y, Z form a **Markov chain** $X \to Y \to Z$.

For instance, Y is a "noisy" measurement of X, and $Z = f(Y)$ is the outcome of deterministic data processing performed on Y, then we have $X \to Y \to Z$.

This implies that

$$I(X\,;\,Z \mid Y) = H(Z \mid Y) - H(Z \mid Y, X) = 0\ .$$

When Y is known, Z doesn't give any extra information about X (and vice versa).

Data Processing Inequality

Assuming that $X \to Y \to Z$ is a Markov chain, we get

$$\begin{aligned} I(X\ ;\ Y, Z) &= I(X\ ;\ Z) + I(X\ ;\ Y \mid Z) \\ &= I(X\ ;\ Y) + I(X\ ;\ Z \mid Y)\ . \end{aligned}$$

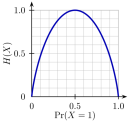
Now, because $I(X\ ;\ Z \mid Y) = 0$, and $I(X\ ;\ Y \mid Z) \geq 0$, we obtain:

Data Processing Inequality

If $X \to Y \to Z$ is a Markov chain, then we have

$$I(X\ ;\ Z) \leq I(X\ ;\ Y)\ .$$

No data-processing can increase the amount of information that we have about X.


1.0
0.5
0
$H(X)$
0
0.5
1.0
$\Pr(X = 1)$

AEP

If $X_1, X_2, \ldots$ is a sequence of *independent and identically distributed* (i.i.d.) r.v.'s with domain $\mathcal{X}$ and pmf p_X, then

$$\log_2 \frac{1}{p_X(X_1)}, \log_2 \frac{1}{p_X(X_2)}, \ldots$$

is also an i.i.d. sequence of r.v.'s.

The expected values of the elements of the above sequence are all equal to the entropy:

$$E\left[\log_2 \frac{1}{p_X(X_i)}\right] = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{1}{p_X(x)} = H(X) \quad \text{for all } i \in \mathbb{N}.$$

AEP

The i.i.d. assumption is equivalent to

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_X(x_i) \quad .$$

AEP

The i.i.d. assumption is equivalent to

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The i.i.d. assumption is equivalent to

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AEP

The i.i.d. assumption is equivalent to

$$\frac{1}{n}\log_2\frac{1}{p(x_1,\ldots,x_n)} = \frac{1}{n}\sum_{i=1}^{n}\log_2\frac{1}{p_X(x_i)} \quad .$$

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$$\frac{1}{n}\log_2\frac{1}{p(x_1,\ldots,x_n)} = \frac{1}{n}\sum_{i=1}^{n}\log_2\frac{1}{p_X(x_i)} \quad .$$

By the (weak) law of large numbers, the average on the right-hand side converges in probability to its mean, i.e., the entropy:

$$\lim_{n\to\infty}\Pr\left[\left|\frac{1}{n}\sum_{i=1}^{n}\log_2\frac{1}{p_X(X_i)} - H(X)\right| < \epsilon\right] = 1 \quad \text{for all } \epsilon > 0.$$

AEP

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$$\frac{1}{n}\log_2 \frac{1}{p(x_1,\ldots,x_n)} = \frac{1}{n}\sum_{i=1}^{n}\log_2 \frac{1}{p_X(x_i)} \quad .$$

Asymptotic Equipartition Property (AEP)

For i.i.d. sequences, we have

$$\lim_{n\to\infty} \Pr\left[\left|\frac{1}{n}\log_2 \frac{1}{p(x_1,\ldots,x_n)} - H(X)\right| < \epsilon\right] = 1$$

for all $\epsilon > 0$.

AEP

The AEP states that for any $\epsilon > 0$, and large enough n, we have

$$\Pr\left[\underbrace{\left|\frac{1}{n}\log_2\frac{1}{p(x_1,\ldots,x_n)} - H(X)\right| < \epsilon}\right] \approx 1$$

$$H(X) - \epsilon < \frac{1}{n}\log_2\frac{1}{p(x_1,\ldots,x_n)} < H(X) + \epsilon$$

AEP

The AEP states that for any $\epsilon > 0$, and large enough n, we have

$$\Pr\left[\underbrace{\left|\frac{1}{n}\log_2\frac{1}{p(x_1,\ldots,x_n)} - H(X)\right| < \epsilon}\right] \approx 1$$

$$n(H(X)-\epsilon) < \log_2\frac{1}{p(x_1,\ldots,x_n)} < n(H(X)+\epsilon)$$

AEP

The AEP states that for any $\epsilon > 0$, and large enough n, we have

$$\Pr\left[\underbrace{\left|\frac{1}{n}\log_2\frac{1}{p(x_1,\ldots,x_n)} - H(X)\right| < \epsilon}\right] \approx 1$$

$$2^{n(H(X)-\epsilon)} < \frac{1}{p(x_1,\ldots,x_n)} < 2^{n(H(X)+\epsilon)}$$

AEP

The AEP states that for any $\epsilon > 0$, and large enough n, we have

$$\Pr\left[\underbrace{\left|\frac{1}{n}\log_2 \frac{1}{p(x_1,\ldots,x_n)} - H(X)\right| < \epsilon}\right] \approx 1$$

$$2^{-n(H(X)+\epsilon)} < p(x_1,\ldots,x_n) < 2^{-n(H(X)-\epsilon)}$$

$$\Leftrightarrow \quad \Pr\left[p(x_1,\ldots,x_n) = 2^{-n(H(X)\pm\epsilon)}\right] \approx 1 \enspace .$$

Asymptotic Equipartition Property (informally)

"Almost all sequences are almost equally likely."

AEP

Technically, the key step in the proof was using the weak law of large numbers to deduce

$$\lim_{n\to\infty} \Pr\left[\left|\frac{1}{n}\sum_{i=1}^{n} \log_2 \frac{1}{p_X(X_i)} - H(X)\right| < \epsilon\right] = 1 \quad \text{for all } \epsilon > 0.$$

In other words, with high probability the average "surprisingness" $\log_2 p_X(X_i)$ over the sequence is close to its expectation.

Typical Sets

Typical Set

The **typical set** $A_\epsilon^{(n)}$ is the set of sequences $(x_1, \ldots, x_n) \in \mathcal{X}^n$ with the property:

$$2^{-n(H(X)+\epsilon)} \leq p(x_1, \ldots, x_n) \leq 2^{-n(H(X)-\epsilon)} \ .$$

The AEP states that

$$\lim_{n \to \infty} \Pr\left[X^n \in A_\epsilon^{(n)}\right] = 1 \ .$$

In particular, for any $\epsilon > 0$, and large enough n, we have

$$\Pr\left[X^n \in A_\epsilon^{(n)}\right] > 1 - \epsilon \ .$$

Typical Sets

How many sequences are there in the typical set $A_\epsilon^{(n)}$?

We can use the fact that by definition each sequence has probability *at least* $2^{-n(H(X)+\epsilon)}$.

Since the total probability of all the sequences in $A_\epsilon^{(n)}$ is trivially *at most* 1, there can't be too many of them.

$$
\begin{aligned}
1 &\geq \sum_{(x_1,\ldots,x_n)\in A_\epsilon^{(n)}} p(x_1,\ldots,x_n) \\
&\geq \sum_{(x_1,\ldots,x_n)\in A_\epsilon^{(n)}} 2^{-n(H(X)+\epsilon)} = 2^{-n(H(X)+\epsilon)} \left| A_\epsilon^{(n)} \right| \\
&\Leftrightarrow \left| A_\epsilon^{(n)} \right| \leq 2^{n(H(X)+\epsilon)} \quad .
\end{aligned}
$$

Typical Sets

Is it possible that the the typical set $A_\epsilon^{(n)}$ is very small?

This time we can use the fact that by definition each sequence has probability *at most* $2^{-n(H(X)-\epsilon)}$.

Since for large enough n, the total probability of all the sequences in $A_\epsilon^{(n)}$ is (by the AEP) *at least* $1-\epsilon$, there can't be too few of them.

$$\begin{aligned} 1-\epsilon &< \Pr\left[X^n \in A_\epsilon^{(n)}\right] \\ &\leq \sum_{(x_1,\ldots,x_n)\in A_\epsilon^{(n)}} 2^{-n(H(X)-\epsilon)} = 2^{-n(H(X)-\epsilon)} \left|A_\epsilon^{(n)}\right| \\ &\Leftrightarrow \left|A_\epsilon^{(n)}\right| > (1-\epsilon) 2^{n(H(X)-\epsilon)} \quad . \end{aligned}$$

Typical Sets

So the AEP guarantees that for small ϵ and large n:

1. The typical set $A_\epsilon^{(n)}$ has high probability.
2. The number of elements in the typical set is about $2^{nH(X)}$.

So what?

Typical Sets

So the AEP guarantees that for small ϵ and large n:

1. The typical set $A_\epsilon^{(n)}$ has high probability.
2. The number of elements in the typical set is about $2^{nH(X)}$.

The number of all possible sequences $(x_1, \ldots, x_n) \in \mathcal{X}^n$ of length n is $|\mathcal{X}|^n$.

The maximum of entropy is $\log_2 |\mathcal{X}|$. If $H(X) = \log_2 |\mathcal{X}|$, we obtain

$$\left|A_\epsilon^{(n)}\right| \approx 2^{nH(X)} = 2^{n \log_2 |\mathcal{X}|} = |\mathcal{X}|^n \ ,$$

i.e., the typical set can be as large as the whole set $\mathcal{X}^n$.

Typical Sets

So the AEP guarantees that for small ϵ and large n:

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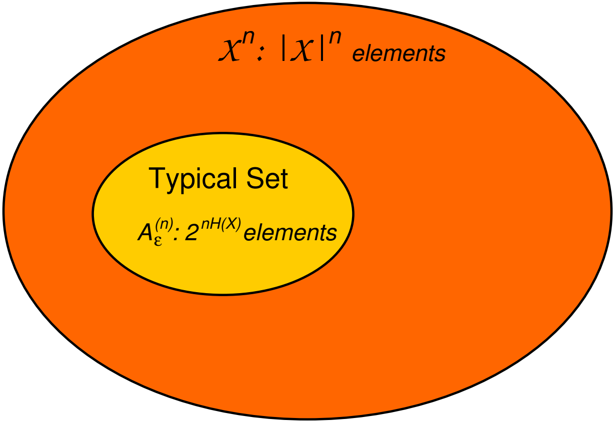
2. The number of elements in the typical set is about $2^{nH(X)}$.

The number of all possible sequences $(x_1, \ldots, x_n) \in \mathcal{X}^n$ of length n is $|\mathcal{X}|^n$.

However, for $H(X) < \log_2 |\mathcal{X}|$, the number of sequences in $A_\epsilon^{(n)}$ is *exponentially smaller* than $|\mathcal{X}|^n$:

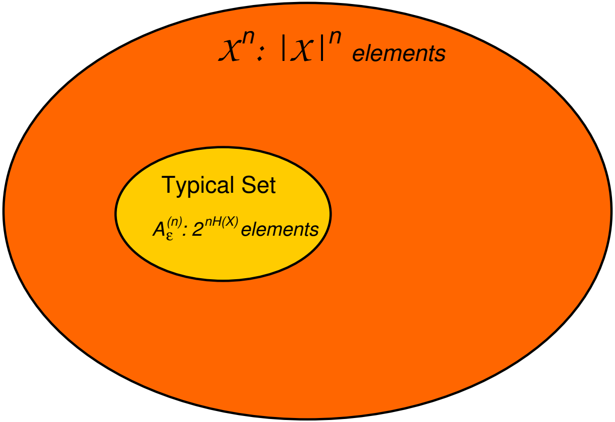
$$\frac{2^{nH(X)}}{2^{n \log_2 |\mathcal{X}|}} = 2^{-n\delta} \xrightarrow[n\to\infty]{} 0 \;, \quad \text{if } \delta = \log_2 |\mathcal{X}| - H(X) > 0.$$

Typical Sets



A (relatively) small set that contains most of the probability mass.

Typical Sets



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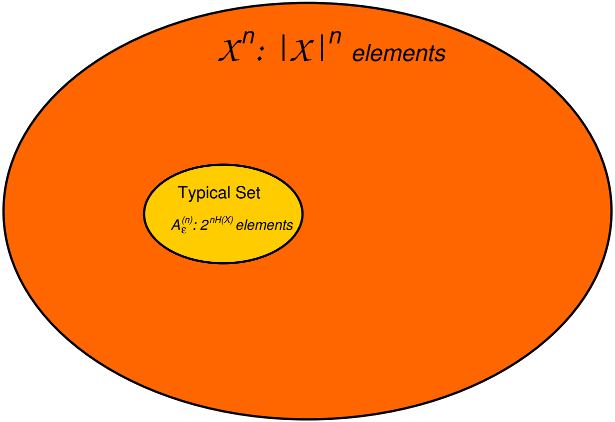

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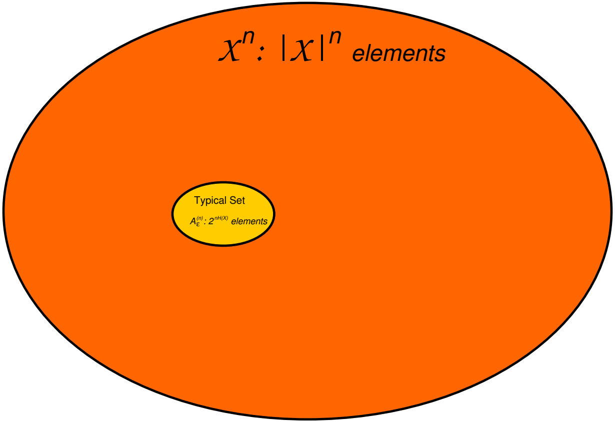

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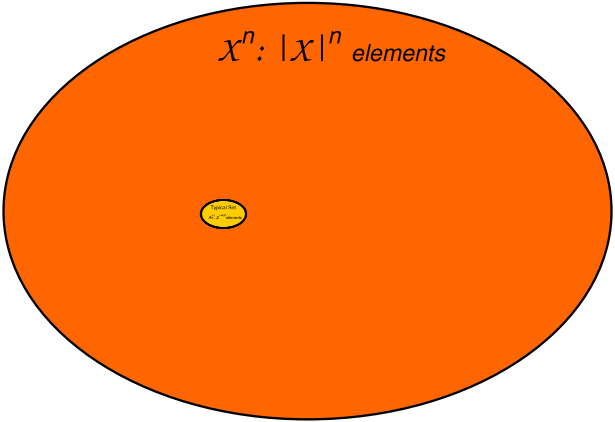

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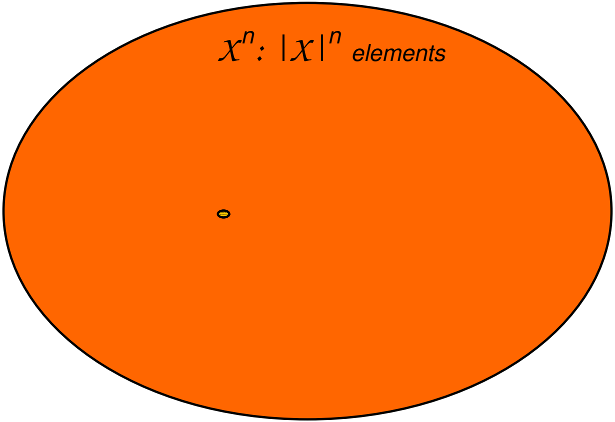

A (relatively) small set that contains most of the probability mass.

Examples

If the source consists of i.i.d. bits $\mathcal{X} = \{0, 1\}$ with $p = p_X(1) = 1 - p_X(0)$, then we have

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_X(x_i) = p^{\sum x_i}(1-p)^{n-\sum x_i} \quad ,$$

where $\sum x_i$ is the number of 1's in x^n.

In this case, the typical set $A_\epsilon^{(n)}$ consists of sequences for which $\sum x_i$ is close to np. For such strings, we have

$$\log_2 \frac{1}{p(x_1, \ldots, x_n)} \approx \log_2 \frac{1}{p^{np}(1-p)^{n(1-p)}}$$
$$= n \left(p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p} \right) = nH(X) \quad .$$

Examples

If the source consists of i.i.d. rolls of a die $\mathcal{X} = \{1, 2, 3, 4, 5, 6\}$ with $p_j = p_X(j)$, $j \in \mathcal{X}$, then we have

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_X(x_i) = \prod_{j=1}^{6} p_j^{k_j} \ ,$$

where k_j is the number of times $x_i = j$ in x^n.

In this case, the typical set $A_\epsilon^{(n)}$ consists of sequences for which k_j is close to np_j for all $j \in \{1, 2, 3, 4, 5, 6\}$. For such strings, we have

$$\log_2 \frac{1}{p(x_1, \ldots, x_n)} \approx \log_2 \frac{1}{\prod_{j=1}^{6} p_j^{np_j}}$$

$$= n \left(\sum_{j=1}^{6} p_j \log \frac{1}{p_j} \right) = nH(X) \ .$$

The AEP Code

We now construct a code from source strings $(x_1, \ldots, x_n) \in \mathcal{X}^n$ to binary sequences $\{0,1\}^*$ of arbitrary length.

Let $x^n \in \mathcal{X}^n$ denote the sequence $(x_1, \ldots, x_n)$, and let $\ell(x^n)$ denote the length (bits) of the codeword assigned to sequence x^n.

The code we will construct has expected per-symbol codeword length arbitrarily close to the entropy

$$E\left[\frac{1}{n}\ell(x^n)\right] \leq H(X) + \epsilon \ ,$$

for large enough n.

! This is **the best achievable rate** for uniquely decodable codes.

The AEP Code

We treat separately two kinds of source strings $x^n \in \mathcal{X}^n$:

1. the **typical** strings $x^n \in A_\epsilon^{(n)}$, and
2. the **non-typical** strings $x^n \in \mathcal{X}^n \setminus A_\epsilon^{(n)}$.

There are at most $2^{n(H(X)+\epsilon)}$ strings of the first kind. Hence, we can encode them using binary strings of length $n(H(X)+\epsilon)+1$.

There are at most $|\mathcal{X}|^n$ strings of the second kind. Hence we can encode them using binary strings of length $n\log_2|\mathcal{X}|+1$.

Since the decoder must be able to tell which kind of a string it is decoding, we prefix the code by a 0 if $x^n \in A_\epsilon^{(n)}$ or by 1 if not. This adds one more bit in either case.

The AEP Code

To see what's going on, consider the situation $H(X) < \log_2 |\mathcal{X}|$. This is the interesting case in which the code actually does result in compression.

In the first lecture we saw that any attempt to compress *everything* will fail because there are not enough short codewords.

We bypass this by splitting into two cases.

1. **Typical** strings are actually compressed. There are not too many of them, so there are enough short codewords.
2. **Non-typical** strings are not compressed. Because their total *probability* is low (**AEP**), this does not matter too much.

Expected Codelength of the AEP Code

Let us calculate the expected per-symbol codeword length:

$$\begin{aligned}
E[\ell(X^n)] &= E\left[\ell(X^n) \;\middle|\; X^n \in A_\epsilon^{(n)}\right] \Pr\left[X^n \in A_\epsilon^{(n)}\right] \\
&\quad + E\left[\ell(X^n) \;\middle|\; X^n \notin A_\epsilon^{(n)}\right] \Pr\left[X^n \notin A_\epsilon^{(n)}\right] \\
&= (n(H(X)+\epsilon)+2) \Pr\left[X^n \in A_\epsilon^{(n)}\right] \\
&\quad + (n \log_2 |\mathcal{X}| + 2) \Pr\left[X^n \notin A_\epsilon^{(n)}\right] \\
&\leq n(H(X)+\epsilon) + n \log |\mathcal{X}| \epsilon + 2 \quad \text{(AEP)} \\
&= n(H(X)+\epsilon') \;,
\end{aligned}$$

where $\epsilon' = \epsilon + \epsilon \log_2 |\mathcal{X}| + \frac{2}{n}$ can be made arbitrarily small by choosing $\epsilon > 0$ small enough, and letting n become large enough.

Optimality of the AEP Code

Dividing this bound by n gives the expected per-symbol codelength of the "AEP code":

$$E\left[\frac{1}{n}\ell(X^n)\right] \leq H(X) + \epsilon$$

for any $\epsilon > 0$ and n large enough.

Optimality: By AEP, there are about $2^{nH(X)}$ sequences that have probability about $2^{-nH(X)}$. We can assign a codeword shorter than $n(H(X) - \delta)$ to only a proportion of less than $2^{-n\delta}$ of these sequences (by a counting argument), and hence the expected per-symbol codeword length must be about $H(X)$ or more.

Noiseless Source Coding Theorem

These two statements give the

9. THE FUNDAMENTAL THEOREM FOR A NOISELESS CHANNEL

We will now justify our interpretation of H as the rate of generating information by proving that H determines the channel capacity required with most efficient coding.

Theorem 9: Let a source have entropy H (bits per symbol) and a channel have a capacity C (bits per second). Then it is possible to encode the output of the source in such a way as to transmit at the average rate $\frac{C}{H} - \epsilon$ symbols per second over the channel where ϵ is arbitrarily small. It is not possible to transmit at an average rate greater than $\frac{C}{H}$.

(Shannon, 1948)

In the noiseless setting with binary code alphabet, the channel capacity is $C = \log_2 |\{0,1\}| = 1$.

The theorem says that the achievable rates are given by

$$R = \lim_{n\to\infty} \frac{n}{\ell(x^n)} < \frac{1}{H(X)} \quad .$$

Coming next

Next on the course:

1. brief excursion into noisy channel coding
2. source coding in practice: efficient algorithms.