Information-Theoretic Modeling

Lectures 11–12: MDL Principle: Applications

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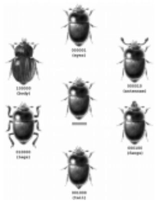
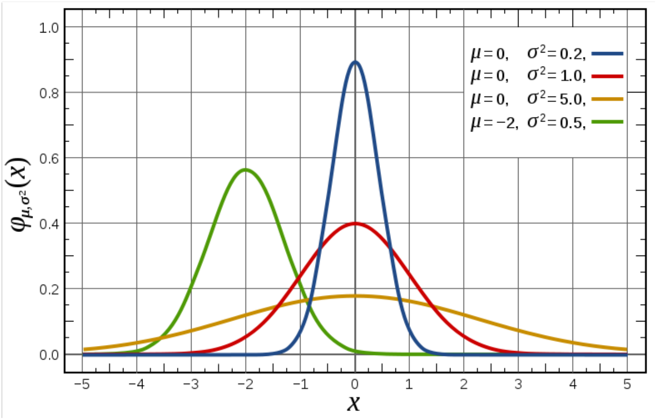


Gaussian models



Source: Wikipedia

Gaussian models

Density function:

$$\phi_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} .$$

Mean: $\mu = E[X]$, variance $\sigma^2 = E[(X-\mu)^2]$

Gaussian models

Density function:

$$\phi_{\mu,\sigma^2}(x_1,\ldots,x_n) \overset{(i.i.d.)}{=} \left(2\pi\sigma^2\right)^{-n/2} e^{-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}} .$$

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Mean: $\mu = E[X]$, variance $\sigma^2 = E[(X-\mu)^2]$

Maximum likelihood: $\hat{\mu} = \frac{1}{n}\sum_{i=1}^n x_i$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \hat{\mu})^2$.

Differential Entropy

What is the optimal rate for encoding (compressing) continuous data?

The answer involves again an entropy. However, not the familiar kind of entropy but instead...

Differential entropy

Let $X \in \mathbb{R}$ be a continuous random variable with probability density $f : \mathbb{R} \to \mathbb{R}^+$.

The differential entropy of X is defined as

$$h(X) = E_{X \sim f}\left[\log_2 \frac{1}{f(X)}\right] = \int f(x) \log_2 \frac{1}{f(x)}\, dx.$$

Differential Entropy

If $\delta > 0$ is small, the probability that $X \in [(t - \frac{1}{2})\delta, (t + \frac{1}{2})\delta]$ is well approximated by $f(t\delta)\delta$.

Hence, the minimum coding rate of the discretized random variable X^δ is given by

$$H(X^\delta) \approx \sum_{x=t\delta\,:\,t\in\mathbb{Z}} f(x)\delta \log_2 \frac{1}{f(x)\delta}$$

$$\xrightarrow[\delta\to 0]{} \int_{-\infty}^{+\infty} f(x) \log_2 \frac{1}{f(x)\delta}\, dx - \log_2 \delta.$$

For finite precision, under regularity conditions, the rate is approximately $H(X^\delta) \approx h(X) - \log_2 \delta$.

Differential Entropy

The differential entropy $h(X)$ measures the uncertainty about a continuous random varible X.

However, its interpretation is less direct than that of $H(X)$.

1. Code-length depends on precision: $H(X^\delta) \approx h(X) - \log_2 \delta$.
2. Item (1) may not be accurate if density $f(x)$ is not smooth enough: $\Pr[x - \delta/2 \leq X < x + \delta/2] \not\approx f(x)\delta$.
3. $H(X) > 0$ but $h(X) \in \mathbb{R}$ (can be negative!)

Mutual information has more familiar properties:

$$I(X\,;\,Y) = h(X) - h(X \mid Y) \geq 0$$

where equality holds only if $X \perp\!\!\!\perp Y$.

Think: $X =$ terrain, $Y =$ map.

$I(\text{map}\,;\,\text{terrain})$

NOVA TOTIVS TERRARVM ORBIS GEOGRAPHICA AC HYDROGRAPHICA TABVLA

Differential Entropy

The minimum coding rate $h(X) - \log_2 \delta$ is achieved if and only if the code-word lengths are chosen according to

$$\ell(x) = \log_2 \frac{1}{f(x)\delta} = \log_2 \frac{1}{f(x)} + \log_2 \frac{1}{\delta} .$$

The term $\log_2(1/\delta)$ depends only on the precision we chose and is same for all models. Therefore, we can ignore it for the purpose of comparing models.

Back to Gaussians

Recall the Gaussian density function:

$$\phi_{\mu,\sigma^2}(x_1,\ldots,x_n) \overset{(i.i.d.)}{=} \left(2\pi\sigma^2\right)^{-n/2} e^{-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}} .$$

The code-length is then

$$\frac{n}{2}\log_2(2\pi\sigma^2) + \frac{1}{(2\ln 2)\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 .$$

Back to Gaussians

Ok, we have our Gaussian code-length formula:

$$\frac{n}{2}\log_2(2\pi\sigma^2)+\frac{1}{(2\ln 2)\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \ .$$

Let's use the two-part code and plug in the maximum likelihood parameters:

$$\hat{\mu}=\frac{1}{n}\sum_{i=1}^{n}x_i, \quad \hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2 \ .$$

Back to Gaussians

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Back to Gaussians

Ok, we have our Gaussian code-length formula:

$$\frac{n}{2}\log_2(2\pi\hat{\sigma}^2)+\frac{n}{2\ln 2} \quad .$$

Let's use the two-part code and plug in the maximum likelihood parameters:

$$\hat{\mu}=\frac{1}{n}\sum_{i=1}^{n}x_i, \quad \hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2 \quad .$$

Back to Gaussians

Ok, we have our Gaussian code-length formula:

$$\frac{n}{2}\log_2 \hat{\sigma}^2 + constant \quad .$$

Let's use the two-part code and plug in the maximum likelihood parameters:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 \quad .$$

Back to Gaussians

We get the total (two-part) code-length formula:

$$\frac{n}{2} \log_2 \hat{\sigma}^2 + \frac{k2}{2} \log_2 n + constant.$$

Since we have two parameters, μ and σ^2, we let $k = 2$.

Notice that depending on what exactly you are doing, you may or may not care about the *constant*.

Regression

A similar treatment can be given to *regression models*.

The model includes one or more regressor variables $x_1, \ldots, x_r \in \mathbb{R}$, and a set of coefficients $\beta_1, \ldots, \beta_p$.

The dependent variable, Y, is assumed to be Gaussian:

- the mean μ is given as a function of the regressors:

$$\mu = f_{\beta_1,\ldots,\beta_p}(x_1, \ldots, x_r),$$

- variance is some parameter σ^2.

Linear Regression

Suppose that the regression function is linear in the parameters.

For a sample of size n, the matrix notation is convenient:

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \quad X = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{np} \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} \quad \epsilon = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

Then the model can be written as

$$Y = X\beta + \epsilon,$$

where $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$.

Linear Regression

FYI: The maximum likelihood estimators are then

$$\hat{\beta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}Y, \quad \hat{\sigma}^2 = \frac{1}{n}\|Y - X\hat{\beta}\|_2^2 = \frac{\mathrm{RSS}}{n},$$

where RSS is the "residual sum of squares".

Since the errors are assumed Gaussian, our code-length formula applies:

$$\frac{n}{2}\log_2 \hat{\sigma}^2\mathrm{RSS} + \frac{kp+1}{2}\log_2 n + \mathrm{constant}.$$

The number of parameters is now $p+1$ (p of the βs and σ^2), so we get...

Nonlinear Regression

The same formula applies also to nonlinear regression models:

$$\frac{n}{2}\log_2 \mathrm{RSS} + \frac{p+1}{2}\log_2 n + \mathrm{constant}.$$

Q: Where does the functional form come from?

- $f(x) = \beta_1 + \beta_2 \sin(\sqrt{\beta_3 x})$?
- $f(x) = \dfrac{\exp(\beta_1 x_1 + \beta_2 x_2)}{1 + \exp(\beta_1 x_1 + \beta_2 x_2)}$?
- ...

In other words, where does the model class come from?

A: No answer... There is no complete and principled answer — look at the data, be creative.

Subset Selection Problem

Often we have a large set of potential regressors, some of which may be irrelevant.

The MDL principle can be used to select a subset of them by comparing the total code-lengths:

$$\min_S \left[\frac{n}{2} \log_2 \mathrm{RSS}_S + \frac{|S|+1}{2} \log_2 n \right],$$

where RSS_S is the RSS obtained by using subset S of the regressors. (Usually linear model.)

Wavelet Denoising

One particularly useful way to obtain the regressor (design) matrix is to use **wavelets**.

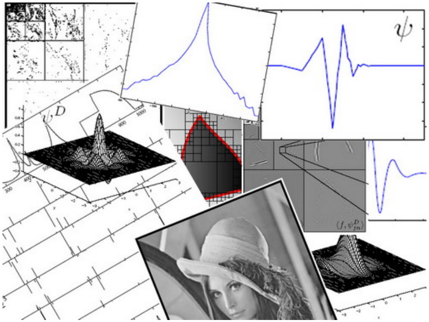
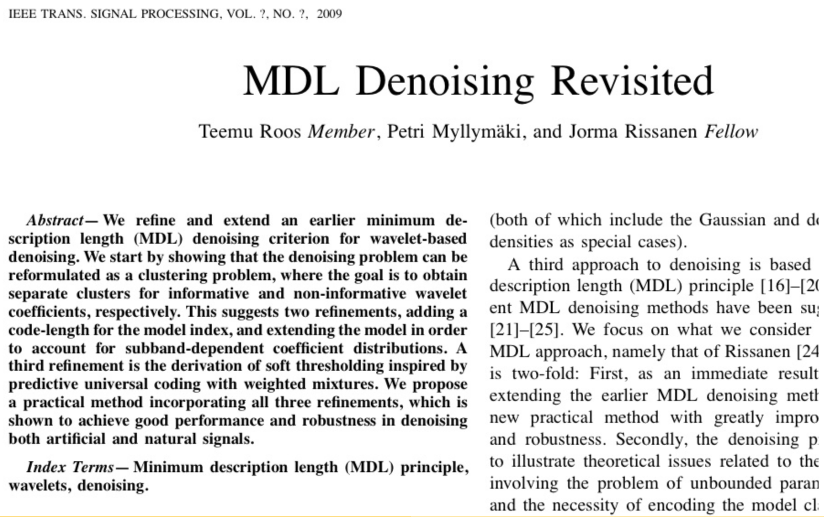


Image by Gabriel Peyré

Wavelet Denoising



IEEE TRANS. SIGNAL PROCESSING, VOL. ?, NO. ?, 2009

MDL Denoising Revisited

Teemu Roos *Member*, Petri Myllymäki, and Jorma Rissanen *Fellow*

Abstract— **We refine and extend an earlier minimum description length (MDL) denoising criterion for wavelet-based denoising. We start by showing that the denoising problem can be reformulated as a clustering problem, where the goal is to obtain separate clusters for informative and non-informative wavelet coefficients, respectively. This suggests two refinements, adding a code-length for the model index, and extending the model in order to account for subband-dependent coefficient distributions. A third refinement is the derivation of soft thresholding inspired by predictive universal coding with weighted mixtures. We propose a practical method incorporating all three refinements, which is shown to achieve good performance and robustness in denoising both artificial and natural signals.**

Index Terms— **Minimum description length (MDL) principle, wavelets, denoising.**

(both of which include the Gaussian and d
densities as special cases).

A third approach to denoising is based
description length (MDL) principle [16]–[2
ent MDL denoising methods have been su
[21]–[25]. We focus on what we consider
MDL approach, namely that of Rissanen [24
is two-fold: First, as an immediate result
extending the earlier MDL denoising meth
new practical method with greatly impro
and robustness. Secondly, the denoising p
to illustrate theoretical issues related to the
involving the problem of unbounded paran
and the necessity of encoding the model cl

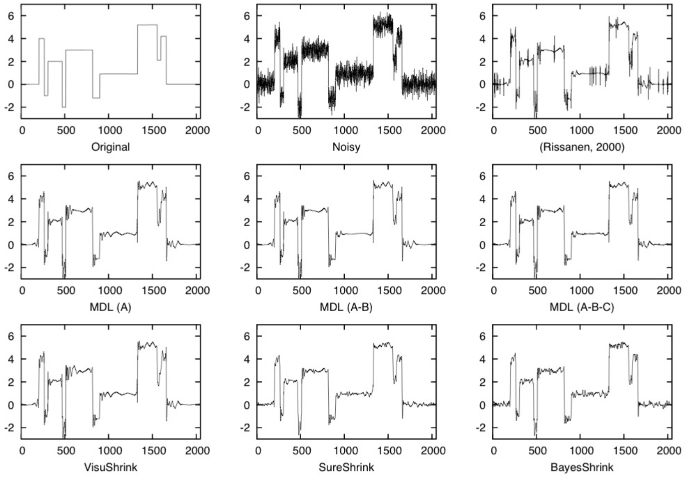
Wavelet Denoising

Linear regression with wavelet basis functions.

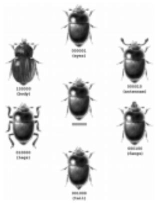
Main effort in constructing a universal code:

1. combines two-part, mixture, and NML universal codes,
2. bounds on NML normalization region required,
3. important lesson: remember to encode model class.

Wavelet Denoising



Wavelet Denoising



Multinomial Models

The multinomial model — the generalization of Bernoulli — is very simple:

$$p(x_j) = \theta_j, \quad \text{for } j \in \{1, \ldots, m\}.$$

Maximum likelihood:

$$\hat{\theta}_j = \frac{\#\{x_i = j\}}{n}.$$

Two-part, mixture, and NML models readily defined.

Fast NML for Multinomials

The naïve way to compute the normalizing constant in the NML model

$$\frac{p_{\hat{\theta}}(x^n)}{C_n^m}, \qquad C_n^m = \sum_{y^n \in \mathcal{X}^n} p_{\hat{\theta}}(y^n),$$

takes exponential time ($\Omega(m^n)$).

The second most naïve way takes “only” polynomial time, $O(n^{m-1})$, but is still intractable unless $m \leq 3$ (or maybe $m \leq 4$).

Fast NML for Multinomials

There is a way — which is not naïve at all! — to do it in linear time, $O(n+m)$, using the following recursion:

$$C_n^m = C_n^{m-1} + \frac{n}{m-2} C_n^{m-2},$$

where C_n^m is the normalizing constant for an m-ary multinomial and sample size n.

The trick is to reduce the general case to $C_n^1 = 1$ and C_n^2, the latter of which can be computed in linear time (using the second most naïve approach).

Kontkanen & Myllymäki, "A linear-time algorithm for computing the multinomial stochastic complexity", *Information Processing Letters* **103** (2007), 6, pp. 227–233

Histogram Density Estimation

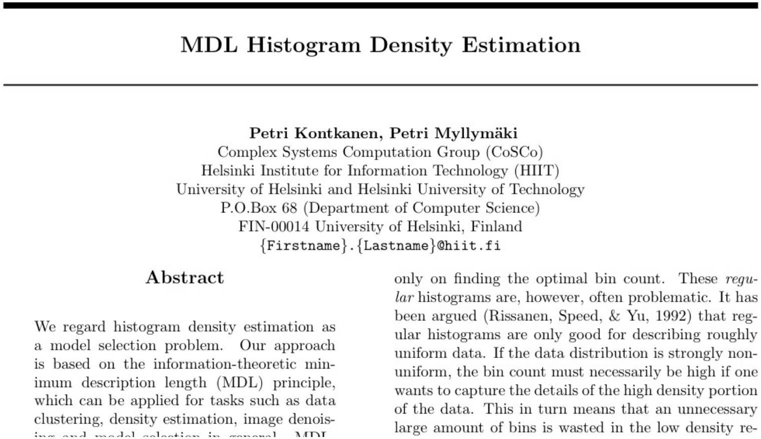
Nonparametric Density Estimation: Given a sample from a univariate density whose parametric form is not known, propose a density estimate.

Histograms are an example of a non-parametric family of densities. Note: Nonparametric = hell of a lot (unbounded number) of parameters!

Choosing the number *and* the positions of break-points in a histogram can be done by MDL.

The code-length is equivalent (up to additive constants) to the code-length in a multinomial model.
$\Rightarrow$ Linear time algorithm can be used.

Histogram Density Estimation



MDL Histogram Density Estimation

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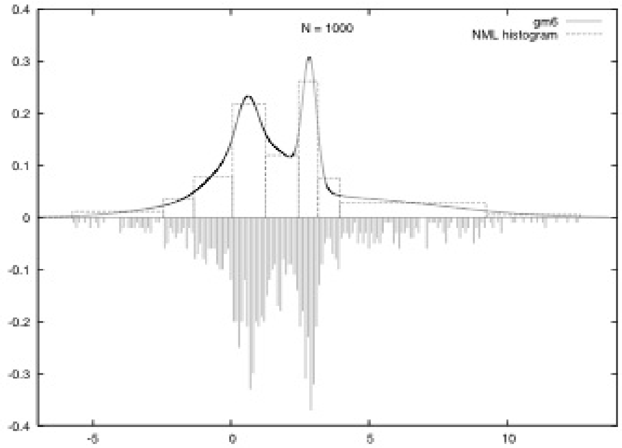
Abstract

We regard histogram density estimation as a model selection problem. Our approach is based on the information-theoretic minimum description length (MDL) principle, which can be applied for tasks such as data clustering, density estimation, image denois-

only on finding the optimal bin count. These *regular* histograms are, however, often problematic. It has been argued (Rissanen, Speed, & Yu, 1992) that regular histograms are only good for describing roughly uniform data. If the data distribution is strongly non-uniform, the bin count must necessarily be high if one wants to capture the details of the high density portion of the data. This in turn means that an unnecessary large amount of bins is wasted in the low density re-

Histogram Density Estimation



Clustering

Consider the problem of clustering vectors of (independent) multinomial variables.

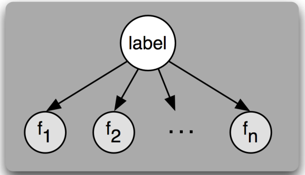
This can be seen as a way to encode (compress) the data:

1. first encode the cluster index of each observation vector,
2. then encode the observations using separate (multinomial) models.

Again, the problem is reduced to the multinomial case, and the fast NML algorithm can be applied.

Clustering

The clustering model can be interpreted as the **naïve Bayes** structure:



label = cluster index $\qquad$ $f_1, \ldots, f_n$ are *features*

The structure is very restrictive. Generalization achieved by **Bayesian networks**.

MDL criterion for learning Bayesian network structures again based on *fast NML for multinomials*.

Coming next

The final week covers some additional topics:

- Kolmogorov complexity
- gambling