The Complexity of Theorem Proving in Autoepistemic Logic

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What is autoepistemic logic?

Autoepistemic Logic

- a non-monotone logic, introduced 1985 by Moore
- models common-sense reasoning
- The language of classical propositional logic is augmented augmented by an unary modal operator L.
- Intuitively, for a formula φ, the formula $L\varphi$ means that φ is believed by a rational agent.

Semantics of autoepistemic logic

The language

$\mathcal{L}^{ae}$ consists of the language $\mathcal{L}$ of classical propositional logic augmented by an unary modal operator L.

Entailment

- An assignment is a mapping from all propositional variables and formulas $L\varphi$ to $\{0, 1\}$.
- For $\Phi \subseteq \mathcal{L}^{ae}$ and $\varphi \in \mathcal{L}^{ae}$, $\Phi \models \varphi$ iff φ is true under every assignment which satisfies all formulas from Φ.
- deductive closure $Th(\Phi) = \{\varphi \in \mathcal{L}^{ae} \mid \Phi \models \varphi\}$.

Semantics of autoepistemic logic

Stable Expansions [Moore 85]

- Informally: a stable expansion corresponds to a possible view of an agent, allowing him to derive all statements of his view from the given premises Σ together with his believes and disbelieves.
- Formally: a stable expansion of $\Sigma \subseteq \mathcal{L}^{ae}$ is a set $\Delta \subseteq \mathcal{L}^{ae}$ satisfying the fixed-point equation

$$\Delta = Th\left(\Sigma \cup \{L\varphi \mid \varphi \in \Delta\} \cup \{\neg L\varphi \mid \varphi \notin \Delta\}\right).$$

Examples

Exactly one expansion

If Σ only consist of objective formulas (no L operators), then the only expansion is the deductive closure of Σ (together with closure under L).

Several expansions

$\{p \leftrightarrow Lp\}$ has two stable expansions:

- one containing p and Lp
- the other containing both $\neg p$ and $\neg Lp$

No expansions

$\{Lp\}$ has no stable expansion.

Two important problems

Credulous Reasoning Problem

Instance: a formula $\varphi \in \mathcal{L}^{ae}$ and a set $\Sigma \subseteq \mathcal{L}^{ae}$

Question: Is there a stable expansion of Σ that includes φ?

Sceptical Reasoning Problem

Instance: a formula $\varphi \in \mathcal{L}^{ae}$ and a set $\Sigma \subseteq \mathcal{L}^{ae}$

Question: Does every stable expansion of Σ include φ?

Previous results

- Semantics and complexity of autoepistemic logic have been intensively studied.
- Credulous Reasoning is Σ_2^p-complete. [Gottlob 92]
- Sceptical Reasoning is Π_2^p-complete. [Gottlob 92]
- Bonatti and Olivetti (ACM ToCL'02) introduced the first purely axiomatic formalism using sequent calculi.

Our results

- We give the first proof-theoretic analysis of the sequent calculi of Bonatti and Olivetti.
- The calculus for credulous autoepistemic reasoning obeys almost the same bounds on the proof size as Gentzen's system LK, i. e., proof lengths are polynomially related.
- For the calculus for sceptical autoepistemic reasoning we show an exponential lower bound to the proof size (even to the number of steps).

The proof systems

Bonatti and Olivetti's sequent calculi for autoepistemic logic consist of three main ingredients:

- classical sequents and rules from LK,
- antisequents to refute non-tautologies,
- autoepistemic rules for the L operator.

Gentzen's LK

Initial sequents: $A \vdash A, \quad 0 \vdash, \quad \vdash 1$

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad \text{(weakening)}$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2} \qquad \text{(exchange)}$$

$$\frac{\Gamma_1, A, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, \Gamma_2 \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta_1, A, A, \Delta_2}{\Gamma \vdash \Delta_1, A, \Delta_2} \qquad \text{(contradiction)}$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \qquad (\neg \text{ introduction})$$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \frac{A, \Gamma \vdash \Delta}{B \wedge A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \quad (\wedge \text{ rules})$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, B \vee A} \quad (\vee \text{ rules})$$

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \quad \text{(cut rule)}$$

The Antisequent Calculus

Axioms: $\Gamma \nvdash \Delta$ where Γ and Δ are disjoint sets of variables.

$$\frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma, \neg\alpha \nvdash \Sigma}\,(\neg \nvdash) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma \nvdash \Sigma, \neg\alpha}\,(\nvdash \neg)$$

$$\frac{\Gamma, \alpha, \beta \nvdash \Sigma}{\Gamma, \alpha \wedge \beta \nvdash \Sigma}\,(\wedge \nvdash) \qquad \frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma \nvdash \Sigma, \alpha \wedge \beta}\,(\nvdash \bullet\wedge) \qquad \frac{\Gamma \nvdash \Sigma, \beta}{\Gamma \nvdash \Sigma, \alpha \wedge \beta}\,(\nvdash \wedge\bullet)$$

$$\frac{\Gamma \nvdash \Sigma, \alpha, \beta}{\Gamma \nvdash \Sigma, \alpha \vee \beta}\,(\nvdash \vee) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma}\,(\bullet\vee \nvdash) \qquad \frac{\Gamma, \beta \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma}\,(\vee\bullet \nvdash)$$

Theorem (Bonatti 93)

The antisequent calculus is sound and complete, i. e., $\Gamma \nvdash \Sigma$ is derivable iff there is an assignment satisfying Γ, but falsifying Σ.

Observation

The antisequent calculus is polynomially bounded.

The credulous autoepistemic calculus

Definition

- A provability constraint is of the form $L\alpha$ or $\neg L\alpha$ with a formula α.
- A set E of formulas satisfies a constraint $L\alpha$ if $\alpha \in Th(E)$.
- Similarly, E satisfies $\neg L\alpha$ if $\alpha \notin Th(E)$.

Definition

- A credulous autoepistemic sequent $\Sigma; \Gamma \mid\!\sim \Delta$ consists of a set Σ of provability constraints, and $\Gamma, \Delta \subseteq \mathcal{L}^{ae}$.
- Semantically, $\Sigma; \Gamma \mid\!\sim \Delta$ is true, if there exists a stable expansion of Γ which satisfies all constraints in Σ and contains $\bigvee \Delta$.

The credulous autoepistemic calculus *CAEL*

Uses rules from LK, the anti-sequent calculus and

$$(\textbf{cA1})\ \frac{\Gamma \vdash \Delta}{;\ \Gamma \mid\!\sim \Delta} \quad (\Gamma \cup \Delta \subseteq \mathcal{L})$$

$$(\textbf{cA2})\ \frac{\Gamma \vdash \alpha \qquad \Sigma;\ \Gamma \mid\!\sim \Delta}{L\alpha,\ \Sigma;\ \Gamma \mid\!\sim \Delta} \quad (\alpha \in \mathcal{L})$$

$$(\textbf{cA3})\ \frac{\Gamma \not\vdash \alpha \qquad \Sigma;\ \Gamma \mid\!\sim \Delta}{\neg L\alpha,\ \Sigma;\ \Gamma \mid\!\sim \Delta} \quad (\Gamma \cup \{\alpha\} \subseteq \mathcal{L})$$

$$(\textbf{cA4})\ \frac{\neg L\alpha,\ \Sigma;\ \Gamma[L\alpha/\bot] \mid\!\sim \Delta[L\alpha/\bot]}{\Sigma;\ \Gamma \mid\!\sim \Delta}$$

$$(\textbf{cA5})\ \frac{L\alpha,\ \Sigma;\ \Gamma[L\alpha/\top] \mid\!\sim \Delta[L\alpha/\top]}{\Sigma;\ \Gamma \mid\!\sim \Delta}$$

In (**cA4**) and (**cA5**) $L\alpha$ is a subformula of $\Gamma \cup \Delta$ and $\alpha \in \mathcal{L}$.

The credulous autoepistemic calculus

Theorem (Bonatti, Olivetti 02)

The calculus is sound and complete, i.e., a credulous autoepistemic sequent is true if and only if it is derivable in CAEL.

The credulous autoepistemic calculus

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Theorem

The length of proofs in CAEL and in LK are polynomially related. The same holds for the number of steps.

More precisely

$s_{LK}(n) \leq s_{CAEL}(n) \leq n(s_{LK}(n) + n^2 + n)$ and
$t_{LK}(n) \leq t_{CAEL}(n) \leq n(t_{LK}(n) + n + 1)$.

where for a proof system P
$s^*_P(x) = \min\{|w| : P(w) = x\}$ and $s_P(n) = \max\{s^*_P(x) : |x| \leq n\}$

A typical derivation

Proofs in *CAEL* are very structured

$$
\cfrac{
  LK/AC \qquad
  \begin{array}{c}
    \cfrac{
      \cfrac{
        LK/AC \qquad \cfrac{LK}{\Gamma' \mid\!\sim \Delta'}\,(\mathbf{cA1})
      }{
        \sigma; \Gamma' \mid\!\sim \Delta'
      }\,(\mathbf{cA2}) \text{ or } (\mathbf{cA3})
    }{
      \vdots
    }\,(\mathbf{cA2}) \text{ or } (\mathbf{cA3}) \\
    \Sigma''; \Gamma' \mid\!\sim \Delta'
  \end{array}
}{
  \begin{array}{c}
    \cfrac{\Sigma'; \Gamma' \mid\!\sim \Delta'}{\vdots}\,(\mathbf{cA4}) \text{ or } (\mathbf{cA5}) \\
    \Sigma; \Gamma \mid\!\sim \Delta
  \end{array}
}\,(\mathbf{cA2}) \text{ or } (\mathbf{cA3})
$$

Sceptical reasoning

Simpler sequents

- Sequents now only consist of two components $\Gamma, \Delta \subseteq \mathcal{L}^{ae}$.
- An *SAEL* sequent is such a pair $\langle \Gamma, \Delta \rangle$, denoted by $\Gamma \mid\!\sim \Delta$.
- Semantically, the *SAEL* sequent $\Gamma \mid\!\sim \Delta$ is true, if $\bigvee \Delta$ holds in all expansions of Γ.

The sceptical autoepistemic calculus *SAEL*

The sceptical autoepistemic calculus uses rules from LK,
the anti-sequent calculus, and

Rules for autoepistemic formulas

(sA1) $\dfrac{\Gamma \vdash \Delta}{\Gamma \mid\!\sim \Delta}$ **(sA2)** $\dfrac{\neg L\alpha, \Gamma \mid\!\sim \alpha}{\neg L\alpha, \Gamma \mid\!\sim \Delta}$ **(sA3)** $\dfrac{L\alpha, \Gamma \not\vdash \alpha}{L\alpha, \Gamma \mid\!\sim \Delta}$

where $\Gamma \cup \{L\alpha\}$ is complete wrt. $ELS(\Gamma \cup \{\alpha\})$ in rule **(sA3)**

(sA4) $\dfrac{L\alpha, \Gamma \mid\!\sim \Delta \qquad \neg L\alpha, \Gamma \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta}$ $(L\alpha \in LS(\Gamma \cup \Delta))$

Theorem (Bonatti, Olivetti 02)

The calculus SAEL is sound and complete, i.e., an SAEL sequent $\Gamma \mid\!\sim \Delta$ *is derivable in SAEL if and only if it is true.*

An exponential lower bound

Theorem

There exist sequents $\Gamma_n \mid\!\sim \Delta_n$ of size $O(n)$ such that every SAEL-proof of $\Gamma_n \mid\!\sim \Delta_n$ has $2^{\Omega(n)}$ steps. Therefore, $s_{SAEL}(n) \in 2^{\Omega(n)}$.

Sketch of proof

- Let

$$\begin{aligned} \Gamma_n &= (p_i \leftrightarrow Lp_i,\ p_i \leftrightarrow q_i)_{i=1,\dots,n} \\ \Delta_n &= \bigwedge_{i=1}^{n} (Lp_i \leftrightarrow Lq_i) \end{aligned}$$

- We will prove that each *SAEL*-proof of $\Gamma_n \mid\!\sim \Delta_n$ contains 2^n applications of rule (**sA4**).
- The antecendent Γ_n has exactly 2^n stable expansions.
- But $\Gamma_n \vdash \Delta_n$ is not classically valid, i.e., not provable in *LK*.

The wider picture

Other non-monotonic logics

- default logic [Reiter 80]
- sequent calculi [Bonatti & Olivetti 02]
- proof-theoretic analysis
  - first-order [Egly & Tompits 01]
  - propositional [B., Meier, Müller, Thomas & Vollmer 11]

Propositional default logic: credulous reasoning

- decision complexity: Σ_2^p-complete [Gottlob 92]
- proof complexity: close link to *LK* [BMMTV 11]

Propositional default logic: sceptical reasoning

- decision complexity: Π_2^p-complete [Gottlob 92]
- proof complexity: exponential lower bound [BMMTV 11]

Attempting an explanation

Proposition

Let L be a language in Σ_2^{p} and let f be any monotone function. Then

- for each propositional proof system P with $s_P(n) \leq f(n)$
- there exists a proof system P' for L with $s_{P'}(n) \leq p(n)f(p(n))$

for some polynomial p.

Our situation

- *LK* corresponds to Bonatti and Olivetti's *CAEL* for autoepistemic logic.
- same correspondence in default logic

Consequently

The sequent calculi of Bonatti and Olivetti for credulous reasoning are as good as one can hope for from a proof complexity perspective.

For sceptical reasoning

Question

Is there a similar connection between propositional proof systems and proof systems for languages in Π_2^p?

One possible approach to proof systems with shorter proofs

- translate autoepistemic formulas into quantified boolean formulas
- use the sequent style calculi of [Krajíček & Pudlák 90] or [Cook & Morioka 05] for QBF
- no lower bounds known for these systems

Summary

Proof complexity for credulous autoepistemic reasoning

is tightly connected to length of proofs in classical logic:

- Bonatti and Olivetti's sequent calculus obeys the same bounds as LK.
- This connection also extends to (non-)automatizability.
- Even holds for stronger proof systems:
  For each propositional proof system we construct a proof system of the same strength for credulous reasoning.

For sceptical autoepistemic reasoning

- we obtain an exponential lower bound.
- Are there better proof systems?