A SAT Approach to Clique-Width

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Motivation

Clique-width is a well-studied in fixed parameter tractability

- over 1200 articles on clique-width on Google Scholar
- small clique-width implies small runtime of various algorithms
- graphs with small clique-width can have arbitrary large tree-width

However, determining the clique-width of a graph is hard

- only very slow algorithms are known
- no existing implementation
- no polynomial-time approximating algorithms
- exact clique-width not known; even for many small graphs

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Contributions

- *Reformulation of Clique-width*
  Developed the concept of a k-derivation of a grpah
- *SAT Encoding of Clique-width*
  An efficient SAT encoding using k-derivations
- *Representative Encoding*
  Arc-consistent encoding for conditional cardinality constraints
- Determined the clique-width of many graphs
  including all graphs up to 10 vertices and famous graphs

Clique-width

Clique-Width

k-graph

A graph whose vertices are labeled by integers from $\{1, \ldots, k\}$.

The *clique-width* of a graph G is the smallest integer k such that G can be constructed from initial k-graphs by means of repeated application of the following three operations.

1. Disjoint union (denoted by $\oplus$);
2. Relabeling: changing all labels i to j (denoted by $\rho_{i \to j}$);
3. Edge insertion: connecting all vertices labeled by i with all vertices labeled by $j, i \neq j$ (denoted by $\eta_{i,j}$ or $\eta_{j,i}$).

Examples

- Cliques (fully connected graphs) have clique-width 2
- Trees have clique-width of at most 3
- An $n \times n$ grid has clique-width $n - 1$

Clique-Width Examples

Examples

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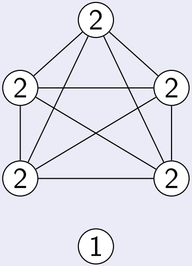
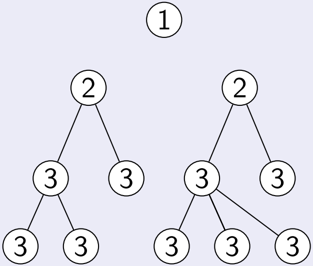
Clique



Tree



Clique-Width into SAT Difficulties

The *clique-width* of a graph G is the smallest integer k such that G can be constructed by repeated application of the following three operations.

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Worst case number of operations

Given a graph $G(V, E)$ the number of operations is in worst case

- Disjoint union: $\mathcal{O}(|V|)$
- Relabeling: $\mathcal{O}(|V|)$
- Edge insertion: $\mathcal{O}(|E|)$ or $\mathcal{O}(|V|^2)$

Reformulation

Templates & Derivations

Reformulation goal: abstract away the edge insertions

Definition (Template)

Given a graph $G = (V, E)$, a *template* T is a partition V into components (induced subgraphs of G) and each component is partitioned into groups (vertices with the same label).

Definition (k-Derivation)

Given a graph $G = (V, E)$, a *k-derivation* of G is a template sequence $(T_0, \dots, T_t)$ with $|cmp(T_0)| = |V|$, $|cmp(T_t)| = 1$, each component in T_i has at most k groups. Furthermore, if there is an edge between two groups in T_i, they must occur in the same component in T_{i-1} and groups can only be merged if they have the same neighborhood with respect to all vertices in the other components.

Example Derivation

Constraint between templates: If there is an edge between two groups, they must occur in the same component before they can be merged.

Merge group constraint: Groups can only be merged if they have the same neighborhood with respect to all vertices in the other components.

A 3-Derivation of a path of length 3: (u)–(v)–(w)–(x)

time	u	v	w	x	template
$t=0$	1	1	1	1	$\{\{\{u\}\},\{\{v\}\},\{\{w\}\},\{\{x\}\}\}$
$t=1$	2 —	1	1	1	$\{\{\{u\},\{v\}\},\{\{w\}\},\{\{x\}\}\}$
$t=2$	3 —	2 —	1	1	$\{\{\{u\},\{v\},\{w\}\},\{\{x\}\}\}$
$t=3$	3 —	3 —	2 —	1	$\{\{\{u,v\},\{w\},\{x\}\}\}$

Encoding

Encoding: Variable and Initial Clauses

Variables:

- $c_{u,v,i}$: vertices $u, v \in V$ are in the same component in template T_i.
- $g_{u,v,i}$: vertices $u, v \in V$ are in the same group in template T_i.

Initial Clauses:

- Initially all vertices are in different components $(\bar{c}_{u,v,0})$
- Eventually all vertices are in the same component $(c_{u,v,t})$
- Vertices in a group are in the same component $(c_{u,v,i} \vee \bar{g}_{u,v,i})$
- Vertices in a component remain in a component $(\bar{c}_{u,v,i-1} \vee c_{u,v,i})$
- Vertices in a group remain in a group $(\bar{g}_{u,v,i-1} \vee g_{u,v,i})$
- Being in a group or in a component is a transitive relation

$$(\bar{c}_{u,v,i} \vee \bar{c}_{v,w,i} \vee c_{u,w,i}) \wedge (\bar{c}_{u,v,i} \vee \bar{c}_{u,w,i} \vee c_{v,w,i}) \wedge (\bar{c}_{u,w,i} \vee \bar{c}_{v,w,i} \vee c_{u,v,i})$$
$$(\bar{g}_{u,v,i} \vee \bar{g}_{v,w,i} \vee g_{u,w,i}) \wedge (\bar{g}_{u,v,i} \vee \bar{g}_{u,w,i} \vee g_{v,w,i}) \wedge (\bar{g}_{u,w,i} \vee \bar{g}_{v,w,i} \vee g_{u,v,i})$$

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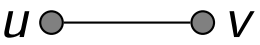
$$(\bar{g}_{u,v,i} \vee \bar{g}_{v,w,i} \vee g_{u,w,i}) \wedge (\bar{g}_{u,v,i} \vee \bar{g}_{u,w,i} \vee g_{v,w,i}) \wedge (\bar{g}_{u,w,i} \vee \bar{g}_{v,w,i} \vee g_{u,v,i})$$

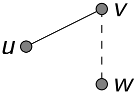
Encoding: Properties



$(c_{u,v,i-1} \vee \bar{g}_{u,v,i})$

Edge Property

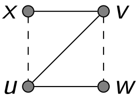
For $u, v \in V$ with $uv \in E$, if u, v are in the same group in T_i, then u, v are in the same component in T_{i-1}.



$(c_{u,v,i-1} \vee \bar{g}_{v,w,i})$

Neighborhood Property

For $u, v, w \in V$ with $uv \in E$ and $uw \notin E$, if v, w are in the same group in T_i, then u, v are in the same component in T_{i-1}.



$(c_{u,v,i-1} \vee \bar{g}_{u,x,i} \vee \bar{g}_{v,w,i})$

Path Property

For $u, v, w, x \in V$, with $uv, uw, vx \in E$ and $wx \notin E$, if u, x and v, w are in the same group in T_i, then u, v are in the same component in T_{i-1}.

Encoding: Direct Encoding of Group Cardinality

Variable $l_{v,j,i}$ denotes that vertex v has group number j in template T_i.

$$\bigwedge_{i \in \{1..t\}} \Big(\bigwedge_{v \in V} (l_{v,1,i} \vee \cdots \vee l_{v,k,i}) \wedge \bigwedge_{u,v \in V} \bigwedge_{j \in \{1..k\}} (\bar{c}_{u,v,i} \vee g_{u,v,i} \vee \bar{l}_{u,j,i} \vee \bar{l}_{v,j,i}) \Big)$$

Example: four vertices $u, v, w, x \in V$ and $k = 3$ (no i for readability)

$$(l_{u,1} \vee l_{u,2} \vee l_{u,3}) \wedge (l_{v,1} \vee l_{v,2} \vee l_{v,3}) \wedge (l_{w,1} \vee l_{w,2} \vee l_{w,3}) \wedge (l_{x,1} \vee l_{x,2} \vee l_{x,3}) \wedge$$
$$(\bar{c}_{u,v} \vee g_{u,v} \vee \bar{l}_{u,1} \vee \bar{l}_{v,1}) \wedge (\bar{c}_{u,v} \vee g_{u,v} \vee \bar{l}_{u,2} \vee \bar{l}_{v,2}) \wedge (\bar{c}_{u,v} \vee g_{u,v} \vee \bar{l}_{u,3} \vee \bar{l}_{v,3}) \wedge$$
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Encoding: Representative and Order Variables

Variable $r_{v,i}$ denotes that v is the representative of its group in T_i.

Vertex v represents group g if and only if for all $u \in g$ holds that $u \geq v$:

$$(r_{v,i} \vee \bigvee_{u \in V, u<v} g_{u,v,i}) \wedge \bigwedge_{u \in V, u<v} (\bar{r}_{v,i} \vee \bar{g}_{u,v,i}) \qquad \text{for } v \in V,\ 0 \leq i \leq t$$

Variable $o^{>}_{v,j,i}$ denotes that the group number of v in T_i is larger than j.

Easy to obtain the group number from order variables

$$l_{v,1,i} = 1 \ \leftrightarrow\ 0000 \ \leftrightarrow\ o^{>}_{v,1,i} = o^{>}_{v,2,i} = o^{>}_{v,3,i} = o^{>}_{v,4,i} = 0$$
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$$l_{v,5,i} = 1 \ \leftrightarrow\ 1111 \ \leftrightarrow\ o^{>}_{v,1,i} = o^{>}_{v,2,i} = o^{>}_{v,3,i} = o^{>}_{v,4,i} = 1$$

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Encoding: Representative Encoding of Group Cardinality

Combining representative and order variables with $u < v$:

$$(\bar{c}_{u,v,i} \vee \bar{r}_{u,i} \vee \bar{r}_{v,i} \vee \bar{o}^{>}_{u,k-1,i}) \wedge (\bar{c}_{u,v,i} \vee \bar{r}_{u,i} \vee \bar{r}_{v,i} \vee o^{>}_{v,1,i}) \wedge$$
$$\bigwedge_{1 \leq a < k-1} (\bar{c}_{u,v,i} \vee \bar{r}_{u,i} \vee \bar{r}_{v,i} \vee \bar{o}^{>}_{u,a,i} \vee o^{>}_{v,a+1,i}) \quad \text{for } u, v \in V,\ 0 \leq i \leq t.$$

Example: four vertices $u, v, w, x \in V$ and $k = 3$ (no i for readability)

$$(\bar{c}_{u,v} \vee \bar{r}_u \vee \bar{r}_v \vee \bar{o}^{>}_{u,2}) \wedge (\bar{c}_{u,v} \vee \bar{r}_u \vee \bar{r}_v \vee o^{>}_{v,1}) \wedge (\bar{c}_{u,v} \vee \bar{r}_u \vee \bar{r}_v \vee \bar{o}^{>}_{u,1} \vee o^{>}_{v,2}) \wedge$$
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Results overview

For all experiments we used the Glucose 2.2 solver. All formulas were generated using the representative encoding of k-derivations.

k	6	7	8	9	10	11	12	13	14
direct	638.5	18,337	TO	TO	TO	TO	30.57	0.67	0.50
repres	12.14	33.94	102.3	358.6	9.21	0.40	0.35	0.32	0.29

To determine the clique-width of a graph $G = (V, E)$, we initialized $k = |V|$ and decreased k until the corresponding formula was unsatisfiable.

Three types of graphs:

- Random graphs with different edge probabilities
- All prime graphs with 10 vertices or less
- Famous graphs

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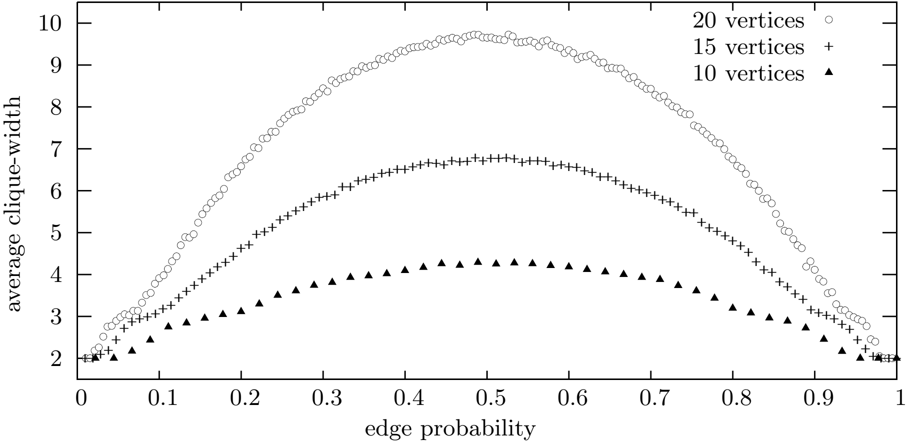
Random Graphs



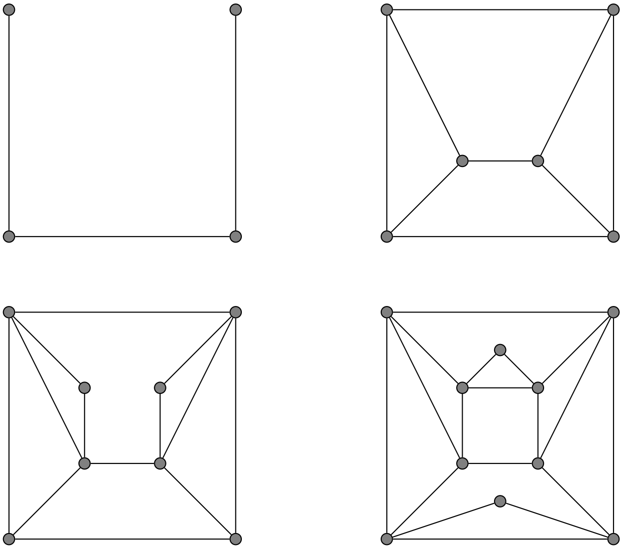
Clique-Width Numbers

			clique-width				
$\|V\|$	connected	prime	2	3	4	5	6
4	6	1	0	**1**	0	0	0
5	21	4	0	4	0	0	0
6	112	26	0	25	**1**	0	0
7	853	260	0	210	50	0	0
8	11,117	4,670	0	1,873	2,790	**7**	0
9	261,080	145,870	0	16,348	125,364	4,158	0
10	11,716,571	8,110,354	0	142,745	5,520,350	2,447,190	**68**

Proposition

The clique-width sequence starts with the numbers 1, 2, 4, 6, 8, 10, 11.

Smallest Graphs with Clique-Width 3, 4, 5, and 6



Famous Graphs

graph	$\|V\|$	$\|E\|$	cwd	variables	clauses	UNSAT	SAT
Brinkmann	21	42	10	8,526	163,065	3,933	1.79
Clebsch	16	40	8	3,872	60,520	191	0.09
Desargues	20	30	8	7,800	141,410	3,163	0.26
Dodecahedron	20	30	8	7,800	141,410	5,310	0.33
Errera	17	45	8	4,692	79,311	82	0.16
Flower snark	20	30	7	8,000	148,620	276	3.90
Folkman	20	40	5	8,280	168,190	12	0.36
Kittell	23	63	8	12,006	281,310	179	18.65
McGee	24	36	8	13,680	303,660	8,700	59.89
Paley-13	13	39	9	1,820	22,776	13	0.05
Paley-17	17	68	11	3,978	72,896	194	0.12
Pappus	18	27	8	5,616	90,315	983	0.14
Robertson	19	38	9	6,422	112,461	478	0.76

Conclusions

Encoded the clique-with problem into SAT

- Conventional formulation is not suitable for encoding
- Reformulation based on derivations enables parallel operations
- Representative encoding is much more efficient than direct encoding

Results

- Discovered the smallest graphs with clique-width 4, 5, and 6
- Observed the influence of the edge-probability on the clique-width
- Determined the clique-width of several famous graphs

Future work

- Evaluate the effectiveness of heuristics for clique-width
- Use the results for theoretical investigations
- Approximating clique-width by limiting the number of steps

A SAT Approach to Clique-Width

Marijn Heule and Stefan Szeider

The University of Texas at Austin, Vienna University of Technology

July 12, 2013 @ SAT