Minimizing Models for Tseitin-Encoded SAT Instances

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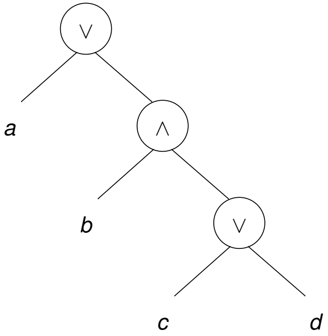

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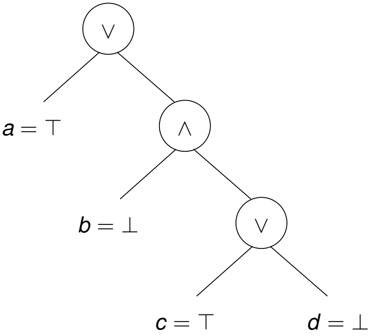

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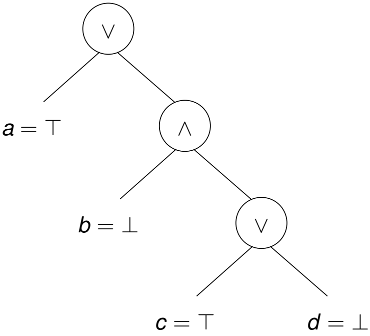
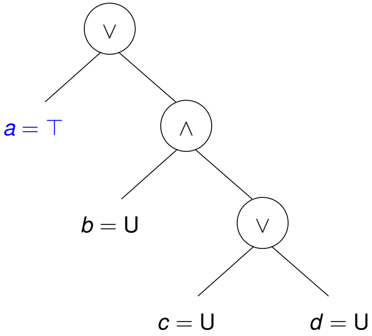


$$M' = \{a\}$$



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Method Overview

Improved minimization of models for Tseitin-encoded formulas by reconstruction of original structure

First Step Naive minimization of models (Hitting Set Problem)

Second Step Minimize w.r.t. input variables

Third Step Reconstruct the original formula's structure and apply the procedure to a subset $F' \subseteq F_{cnf}$

Why minimize?

Shorter description of counterexamples help us to . . .

- boost abstraction-refinement loop in DPLL(T) based SMT solvers
- enhance usability of SAT-based verification tools
- boost model counting

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Minimizing Models for CNF Formulas

$F_{cnf} = \{\{t_0\}, \{\neg t_0, a, t_1\}, \{\neg t_1, b\}, \{\neg t_1, t_2\}, \{\neg t_2, c, d\}\}$

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Determine subset $M' \subseteq M$ such that each clause is satisfied

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Compute a minimal Hitting Set with SAT

Computing a Minimal Hitting Set with SAT

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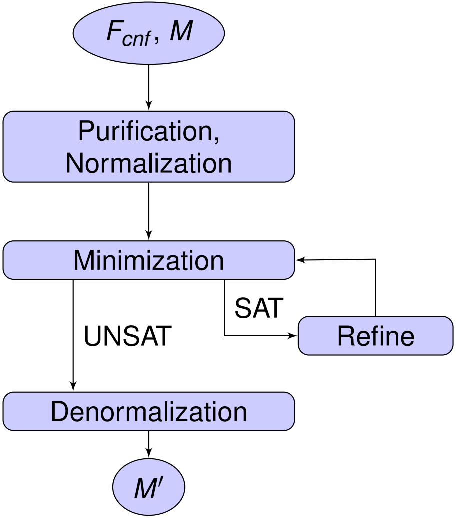
Algorithm Overview



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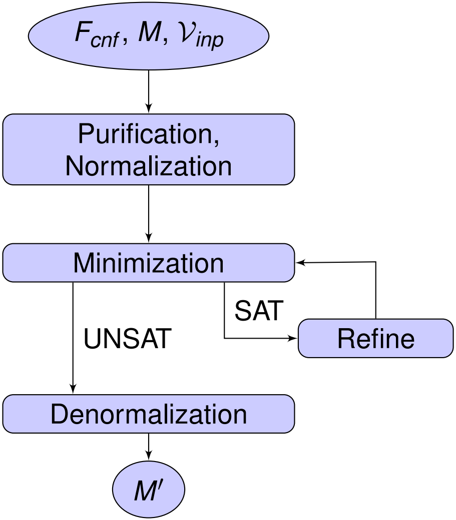

The Effect of the Tseitin Encoding on Model Size

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$F_{cnf} = \{\{t_0\}, \{\neg t_0, a, t_1\}, \{\neg t_1, b\}, \{\neg t_1, t_2\}, \{\neg t_2, c, d\}\}$

Full Model:	$M = \{a, \neg b, c, \neg d, t_0, \neg t_1, t_2\}$
Hitting Set:	$M' = \{a, c, t_0, \neg t_1\}$
Input Variables:	$M'' = \{a, c\}$

- What happened to our shorter model $\{a\}$ for the original formula?
- Is there a way to obtain that shorter model from F_{cnf}?

Analyze Assignment of Encoding Variables

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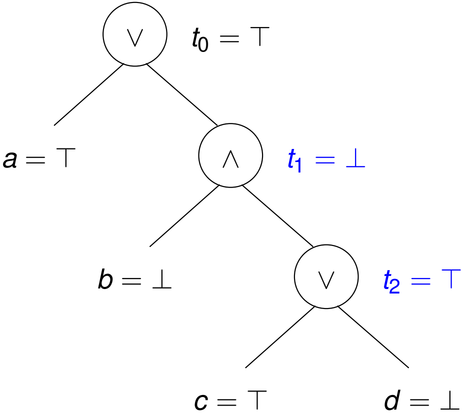

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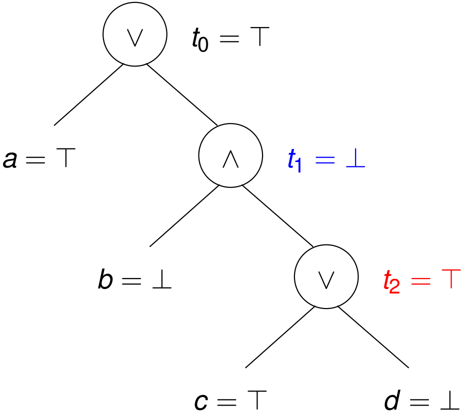

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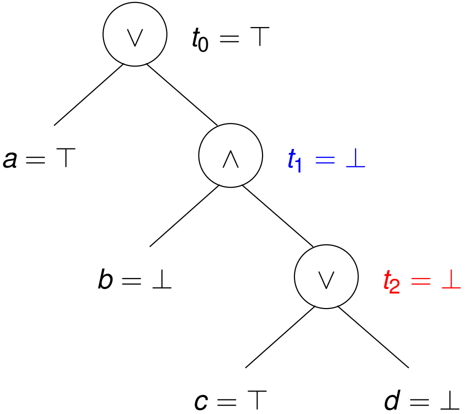

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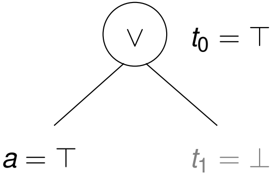

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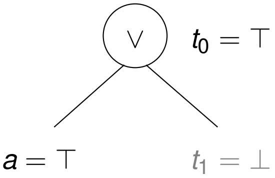

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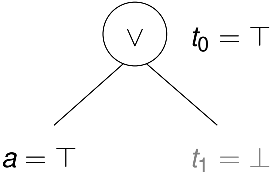

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We can reconstruct the original formula's structure

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Follow the Implication

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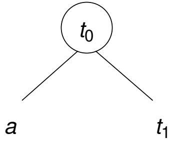

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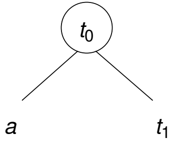

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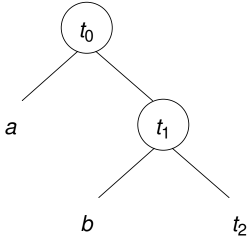

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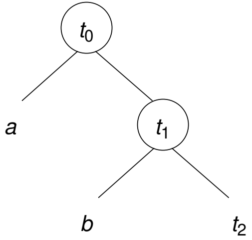

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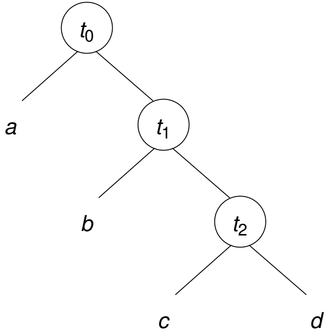

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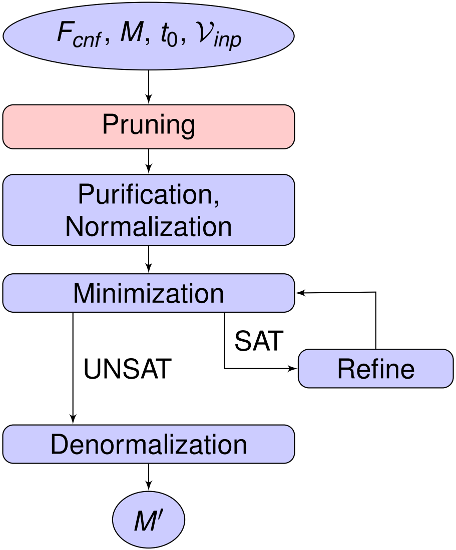
Pruning as a Preprocessing Step



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- Consider different encodings (e.g. only partially Tseitin)
- Model Counting (with an application to Quantitative Information Flow Analysis)

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