A Rank Lower Bound for Cutting Planes Proofs of Ramsey's Theorem

Massimo Lauria
KTH Royal Institute of Technology (Stockholm)

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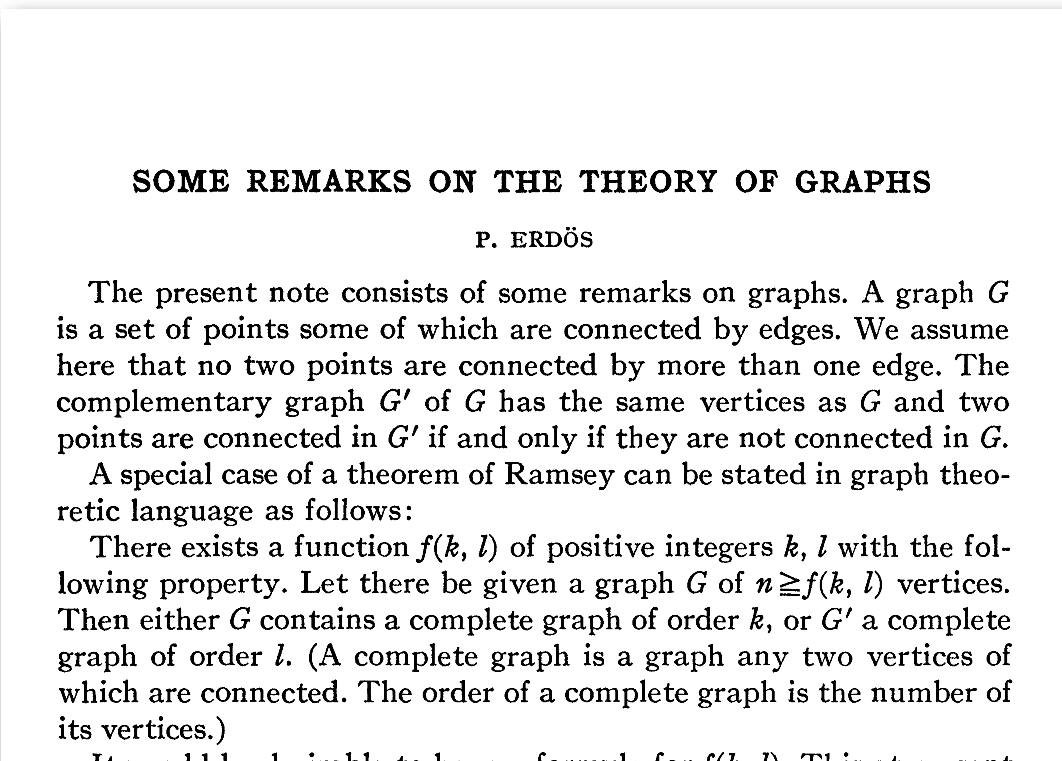
Ramsey Theorem

There is a function R(k) such that any graph with R(k) vertices has either a clique or and independent set of size k.

$$2^{k/2} < R(k) < 4^k$$



SOME REMARKS ON THE THEORY OF GRAPHS

P. ERDÖS

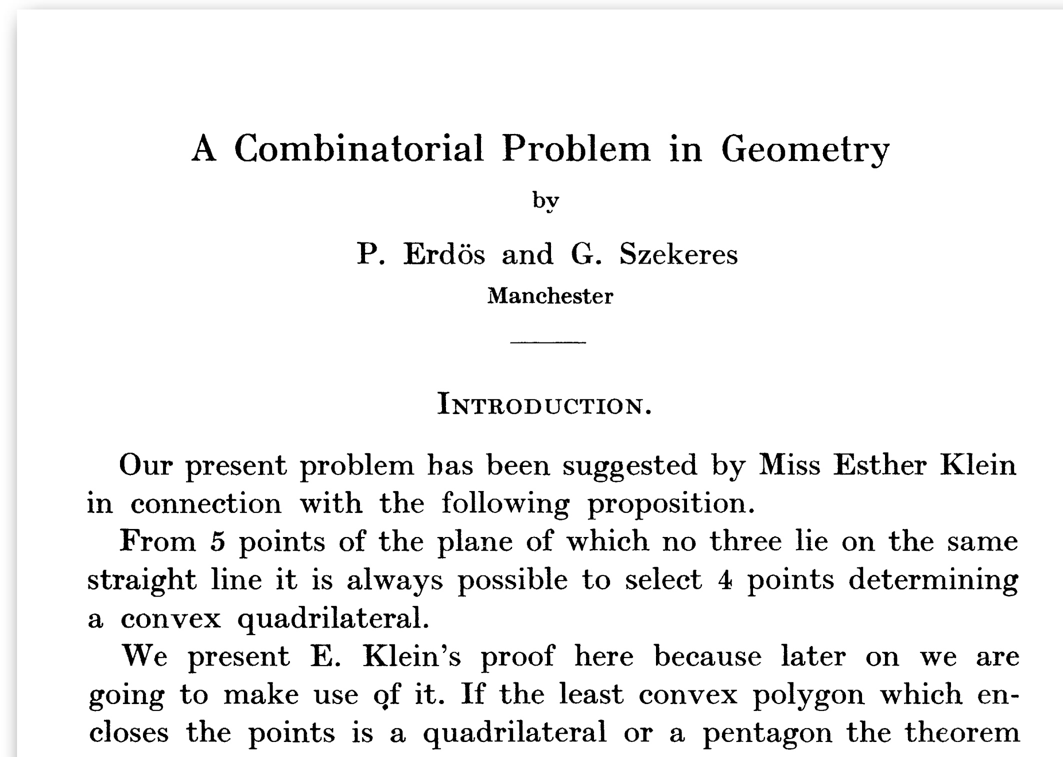
The present note consists of some remarks on graphs. A graph G is a set of points some of which are connected by edges. We assume here that no two points are connected by more than one edge. The complementary graph G' of G has the same vertices as G and two points are connected in G' if and only if they are not connected in G.

A special case of a theorem of Ramsey can be stated in graph theoretic language as follows:

There exists a function $f(k, l)$ of positive integers k, l with the following property. Let there be given a graph G of $n \geqq f(k, l)$ vertices. Then either G contains a complete graph of order k, or G' a complete graph of order l. (A complete graph is a graph any two vertices of which are connected. The order of a complete graph is the number of its vertices.)

It would be desirable to have a formula for $f(k, l)$. This at present

[E47]



A Combinatorial Problem in Geometry

by

P. Erdös and G. Szekeres

Manchester

INTRODUCTION.

Our present problem has been suggested by Miss Esther Klein in connection with the following proposition.

From 5 points of the plane of which no three lie on the same straight line it is always possible to select 4 points determining a convex quadrilateral.

We present E. Klein's proof here because later on we are going to make use of it. If the least convex polygon which encloses the points is a quadrilateral or a pentagon the theorem is trivial. Let therefore the enclosing polygon be a triangle ABC.

[ES35]

Proof complexity of bounding $R(k)$

$R(k) \leq 4^k$ upper bound

- [Pudlák '91] easy in bounded depth sequent calculus
- [Pudlák '12] requires large proofs in resolution

$R(k) \leq n$ upper bound for $n = R(k) + O(1)$

- [Krajíček '11] hard for bounded depth sequent calculus

$R(k) > 2^{k/2}$ lower bound

- [**L.**, Pudlák, Rödl, Thapen '13] requires large proofs in resolution

In this work

We show a lower bound for the "logical depth" (aka **rank**) of proving

$$R(k) \leq 4^k$$

in cutting planes.

Cutting planes proofs model
integer programming techniques

- performance on combinatorial problems
- no lower bound is known for non artificial formulas

↑

this is why we focus on logical depth

Outline

i. cutting planes proofs

ii. logical depth (i.e. rank) as a measure of hardness

iii. lower bound for "$R(k) \leq 4^k$"

1.
Cutting planes

A CNF is turned into a system of inequalities

$$x \vee y \vee \neg z \longrightarrow x + y + (1 - z) \geq 1$$

A refutation is the derivation of

$$-1 \geq 0$$

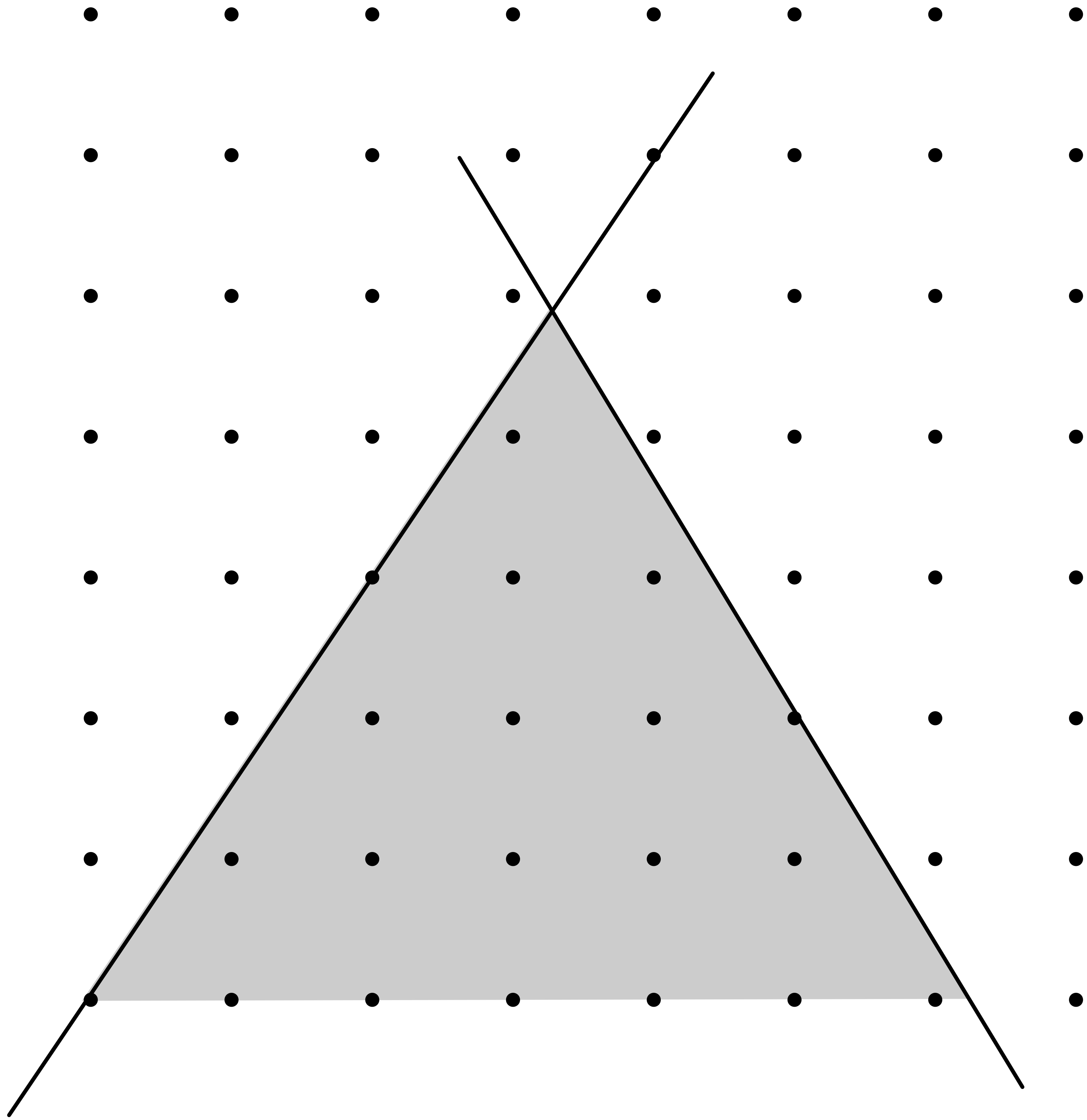
Variables: $x_i \in \{0, 1\}$

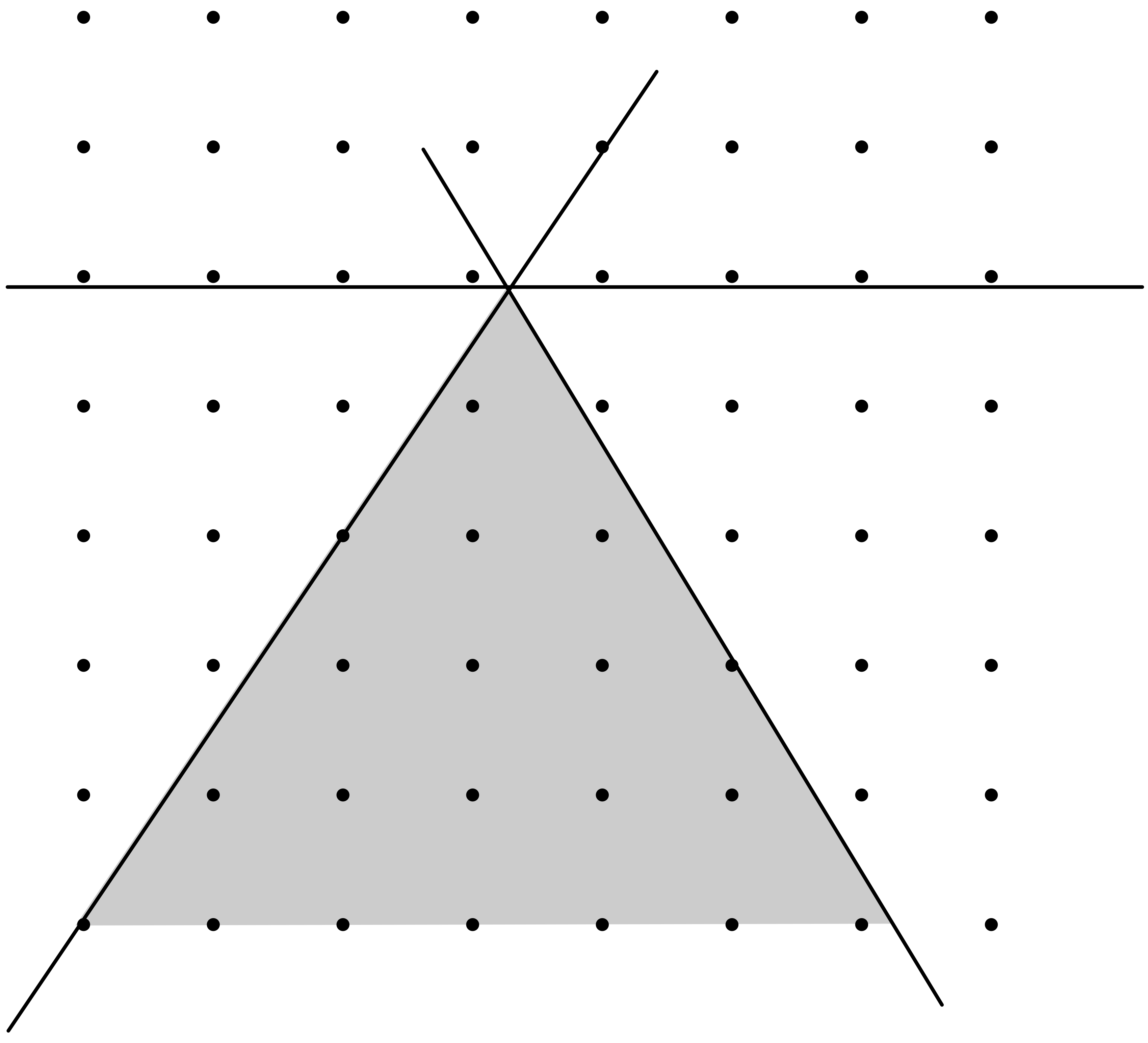
Proof lines: $$a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n \leq b$$

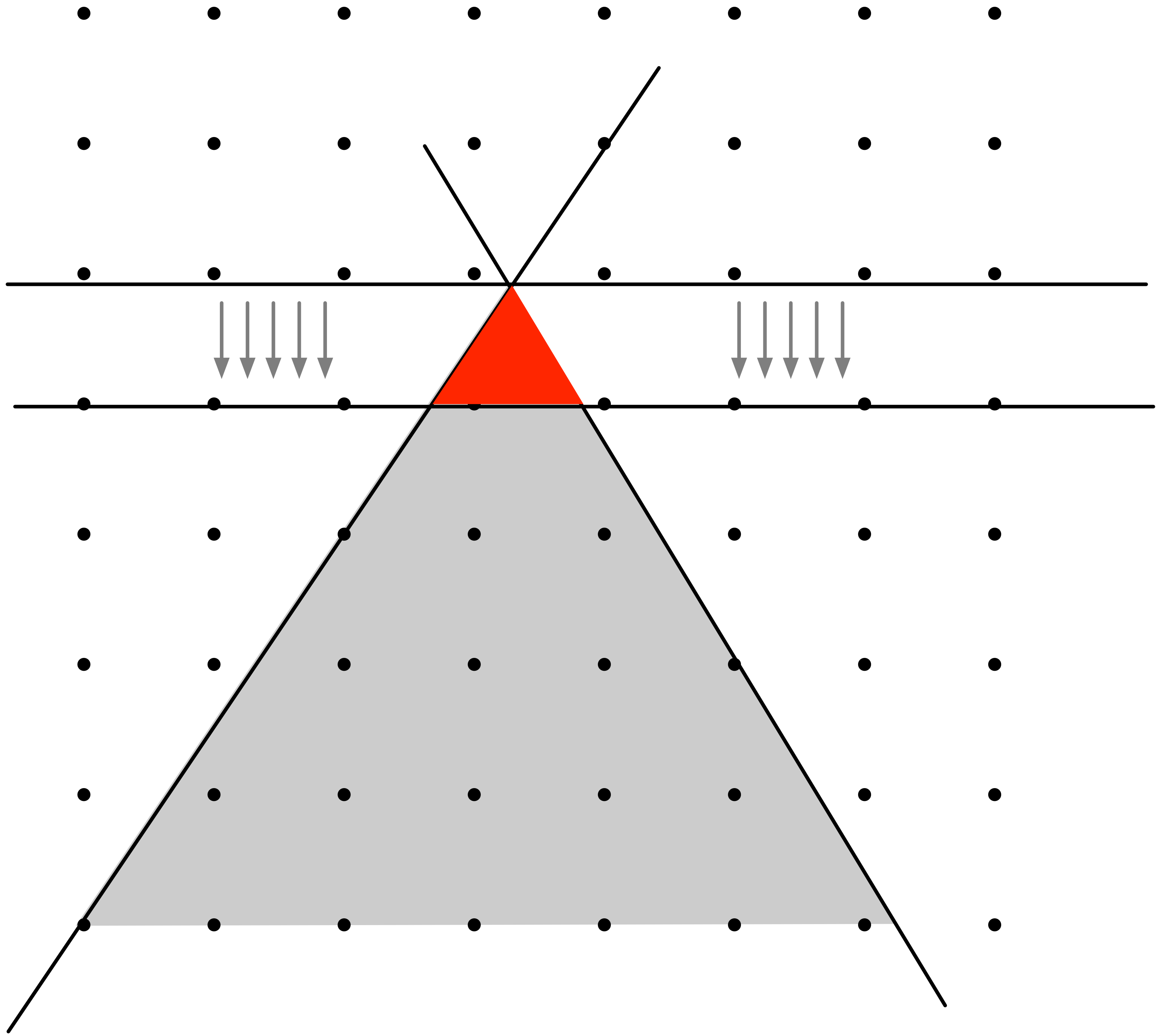
with $a_i \in \mathbb{Z}$ and $b \in \mathbb{Z}$

Sum: $$\frac{\sum a_i x_i \leq b \qquad \sum a'_i x_i \leq b'}{\sum (\alpha a_i + \beta a'_i) x_i \leq \alpha b + \beta b'} \quad \alpha, \beta \in \mathbb{N}$$

Cut: $$\frac{\sum c a_i x_i \leq b}{\sum a_i x_i \leq \lfloor \frac{b}{c} \rfloor} \quad c \in \mathbb{N}$$







Results on cutting planes

- [Pudlák '97] There is a CNF formula with no polynomial length cutting planes refutations.
- [BGHMP '03] Linear rank lower bounds for random 3-CNF and Tseitin formulas.

II.
Rank of a refutation

Initial inequalities have rank 0

$$\frac{\sum a_i x_i \leq b \qquad \sum a'_i x_i \leq b'}{\sum (\alpha a_i + \beta a'_i) x_i \leq \alpha b + \beta b'}$$

$$\frac{\sum c a_i x_i \leq b}{\sum a_i x_i \leq \lfloor \frac{b}{c} \rfloor}$$

Initial inequalities have rank 0

r_1 r_2

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$\max(r_1, r_2)$

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$r + 1$

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r

$$\frac{\sum c a_i x_i \leq b}{\sum a_i x_i \leq \lfloor \frac{b}{c} \rfloor}$$

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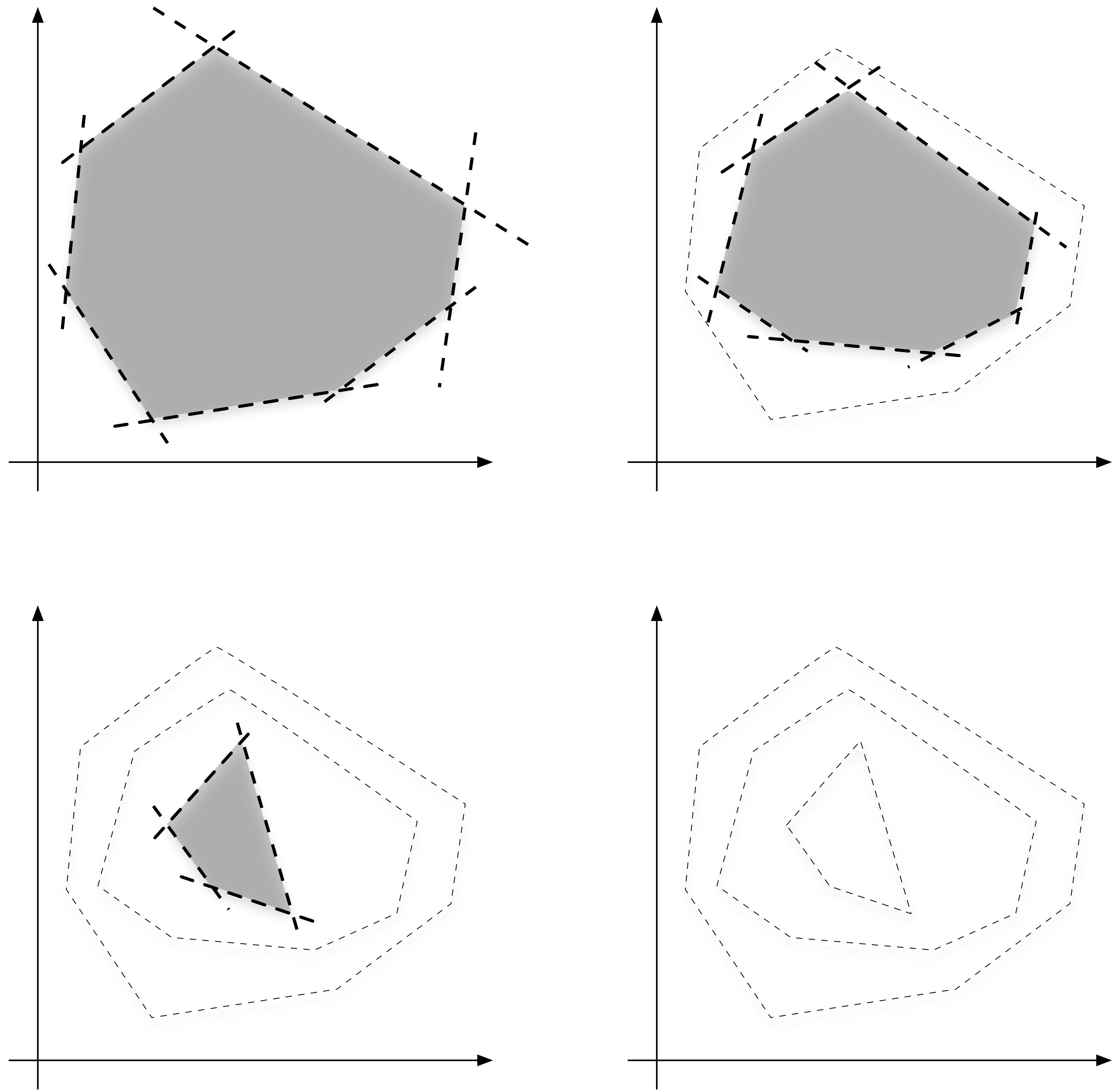
Rank of a refutation: rank of inequality $0 \leq -1$.

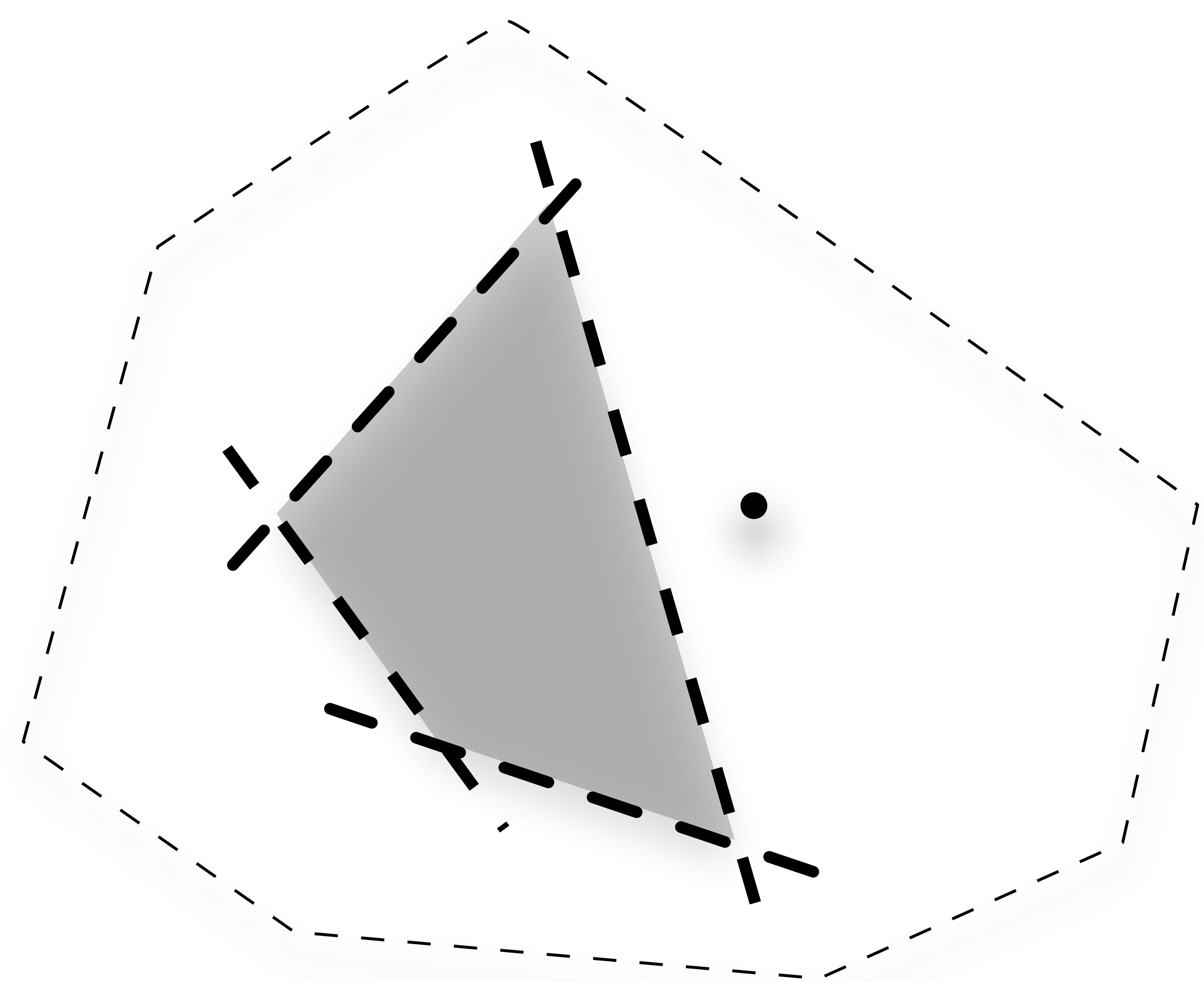
(for CNF it is at most the number of variable)



Thm [CCH'89]: any inequality of rank d can be proved in length $O(n^d)$.

(viceversa does not hold [BGHMP '03])

Rank of a point: is the smallest rank among inequalities which eliminate the point.

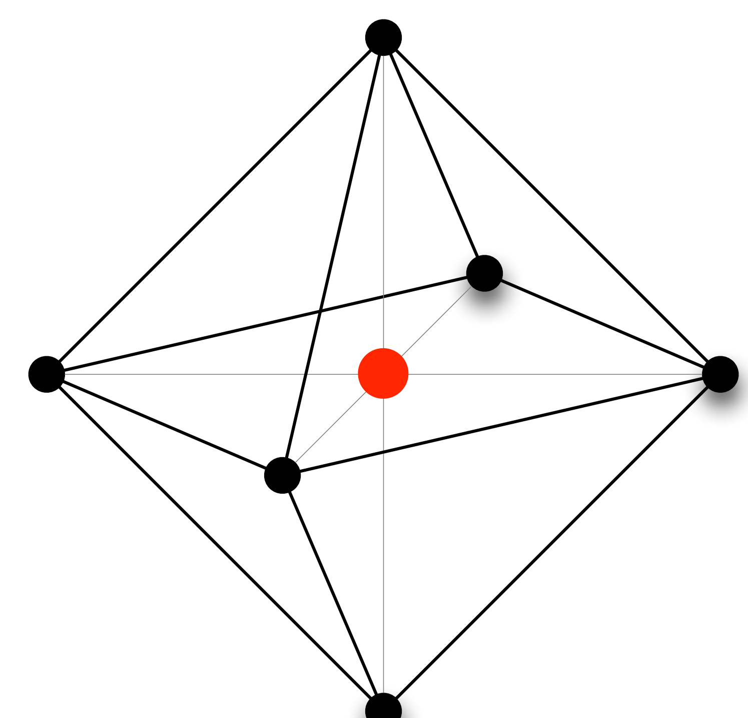


GOAL

Prove that a fractional solution has large rank

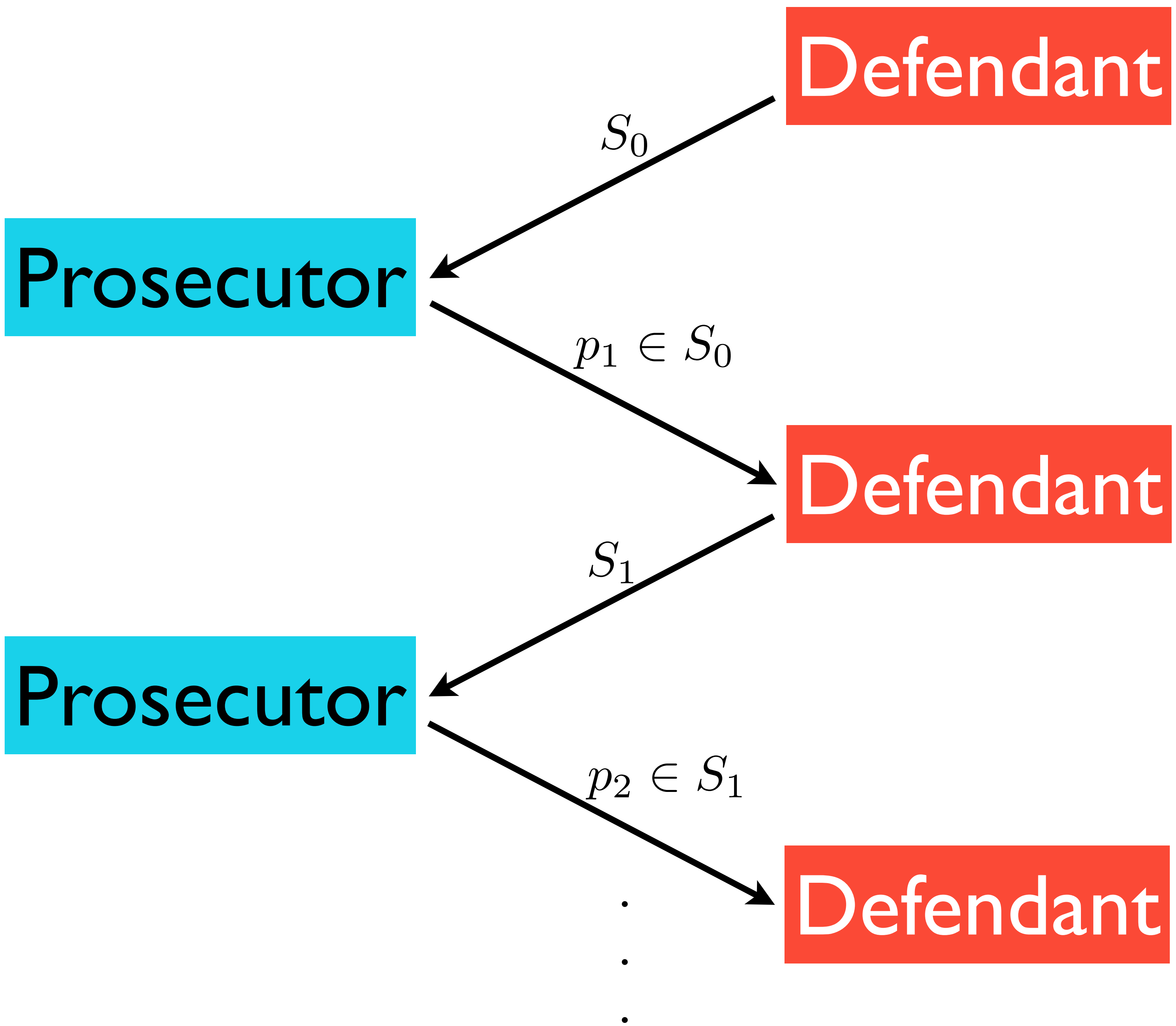
TOOL

Protection lemma: if all points in the "protection set" P for point p have rank at least r, then point p has rank $r+1$.



Start: a feasible point p_0

Each S_i is a protection set of p_i and it is feasible.



If $p_0 \dots p_r$ are always feasible then p_0 has rank $\geq r$

III.

Lower bound for "$R(k) \leq 4^k$"

Encoding the **negation** of $R(k) \leq 4^k$ bound

Fix $V = [4^k]$ and variables $x_e \in \{0,1\}$ for $e \in \binom{V}{2}$

$$\forall S \in \binom{V}{k} \qquad 1 \leq \sum_{e \in \binom{S}{2}} x_e \leq \frac{k(k-1)}{2} - 1$$

The size of the formula is $k^2 4^{k^2} = |V|^{O(\log |V|)}$

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S is not an independent set

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S is not an independent set (lower bound) — S is not a clique (upper bound)

The size of the formula is $k^2 4^{k^2} = |V|^{O(\log |V|)}$

Prosecutor wants to show that $R(k) \leq 4^k$.

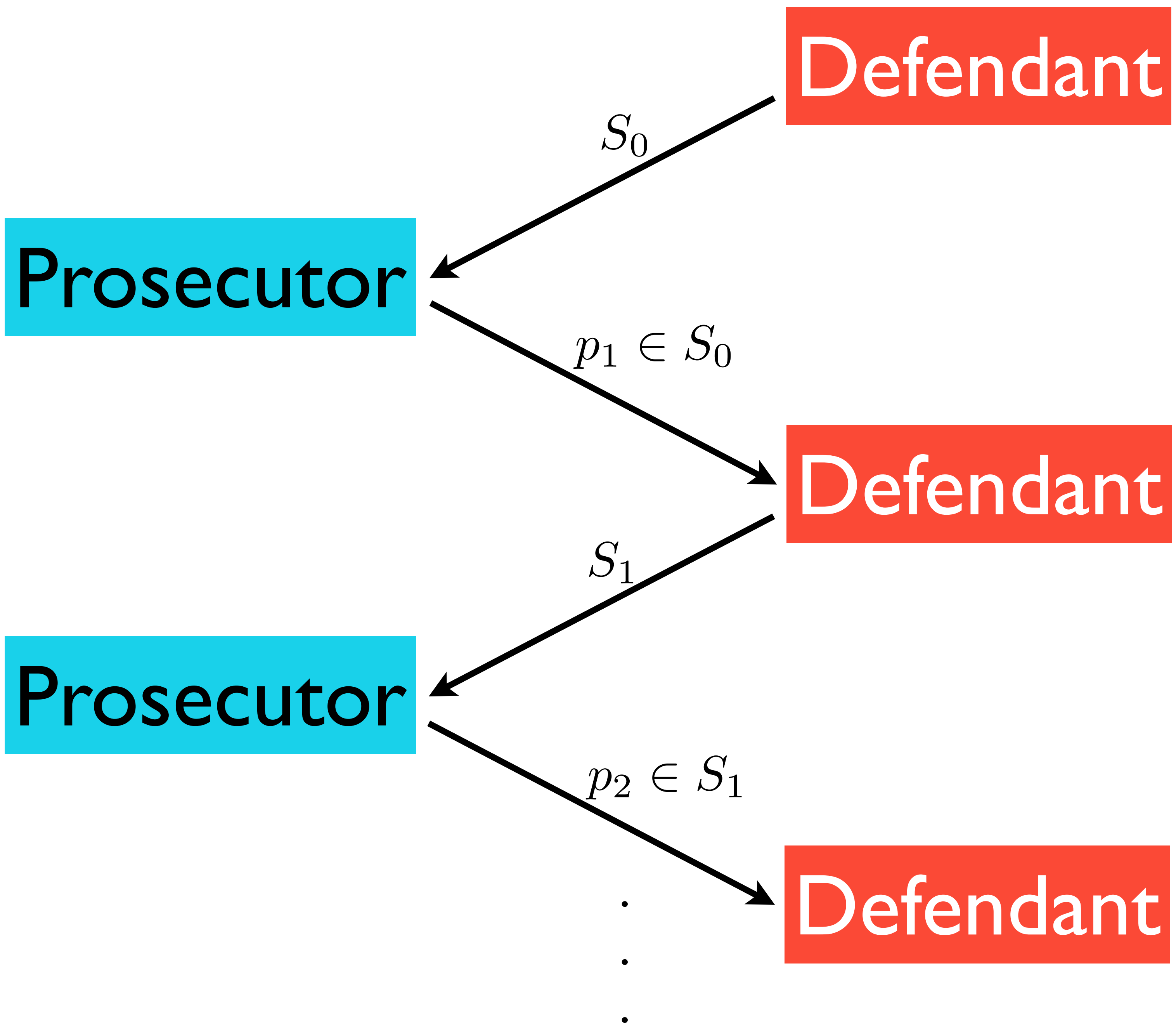
Defender uses a model graph with no k clique/ind. set

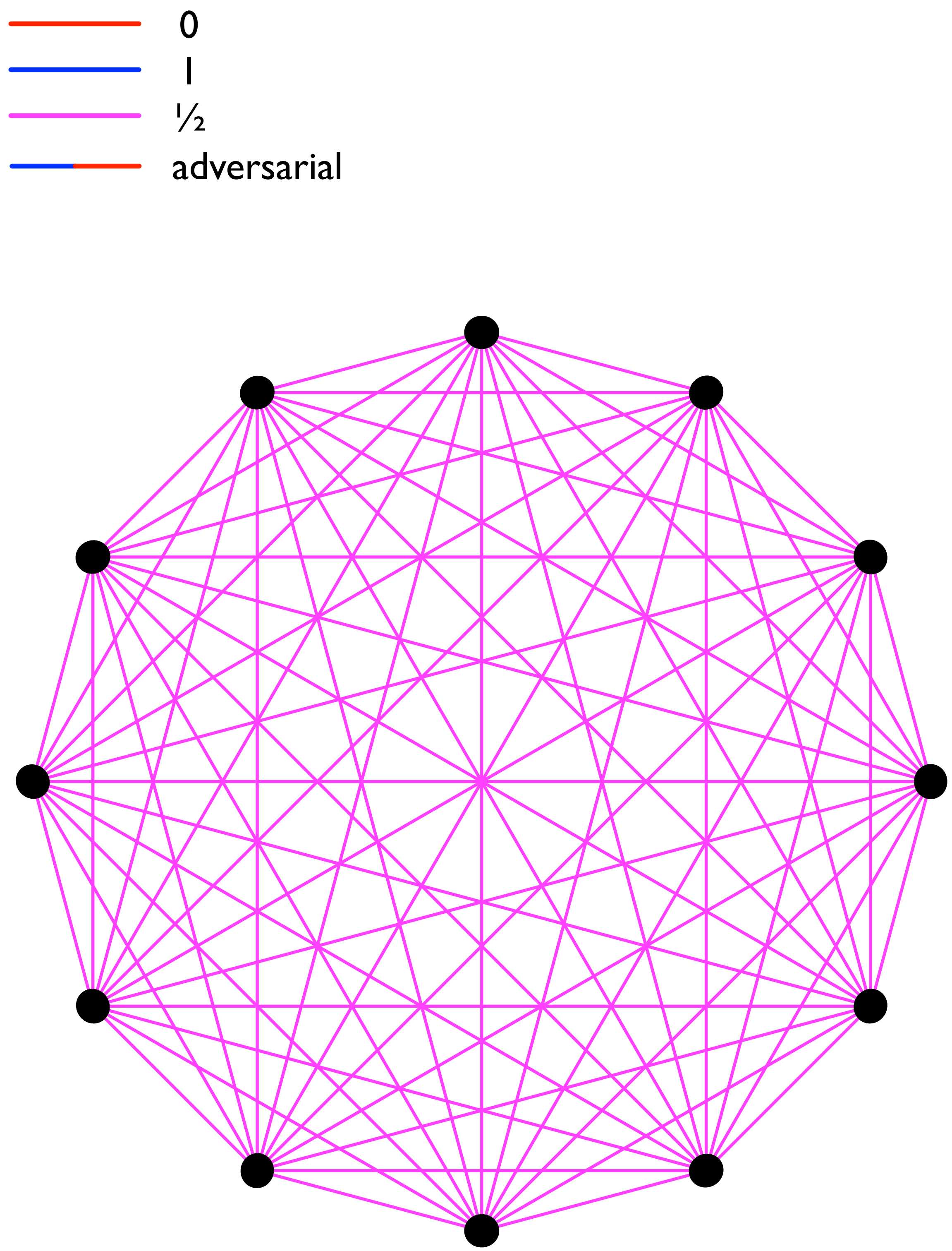
as big as possible

in order to fool the Prosecutor

as long as possible.

$$p_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots \frac{1}{2}, \frac{1}{2})$$





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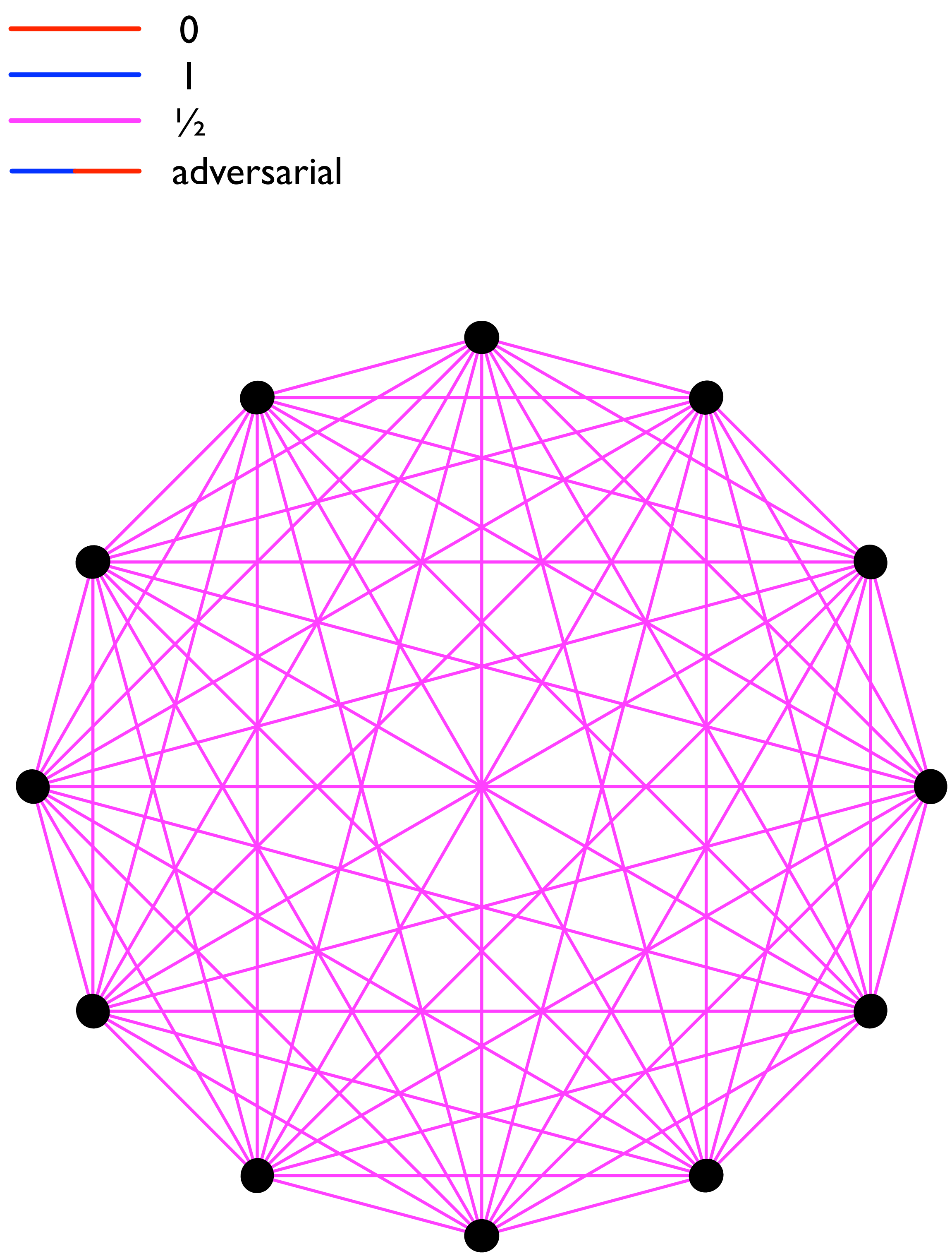
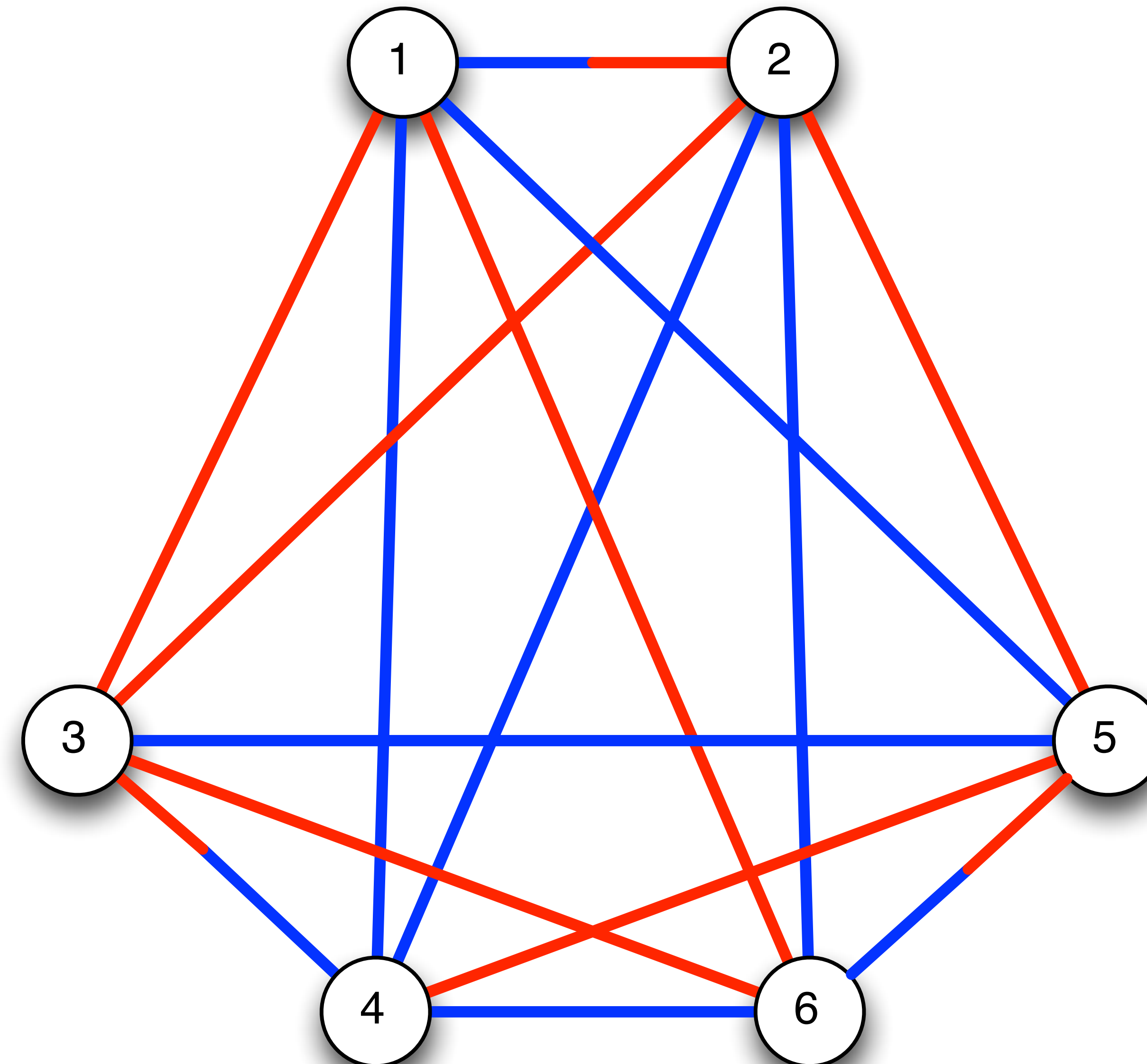
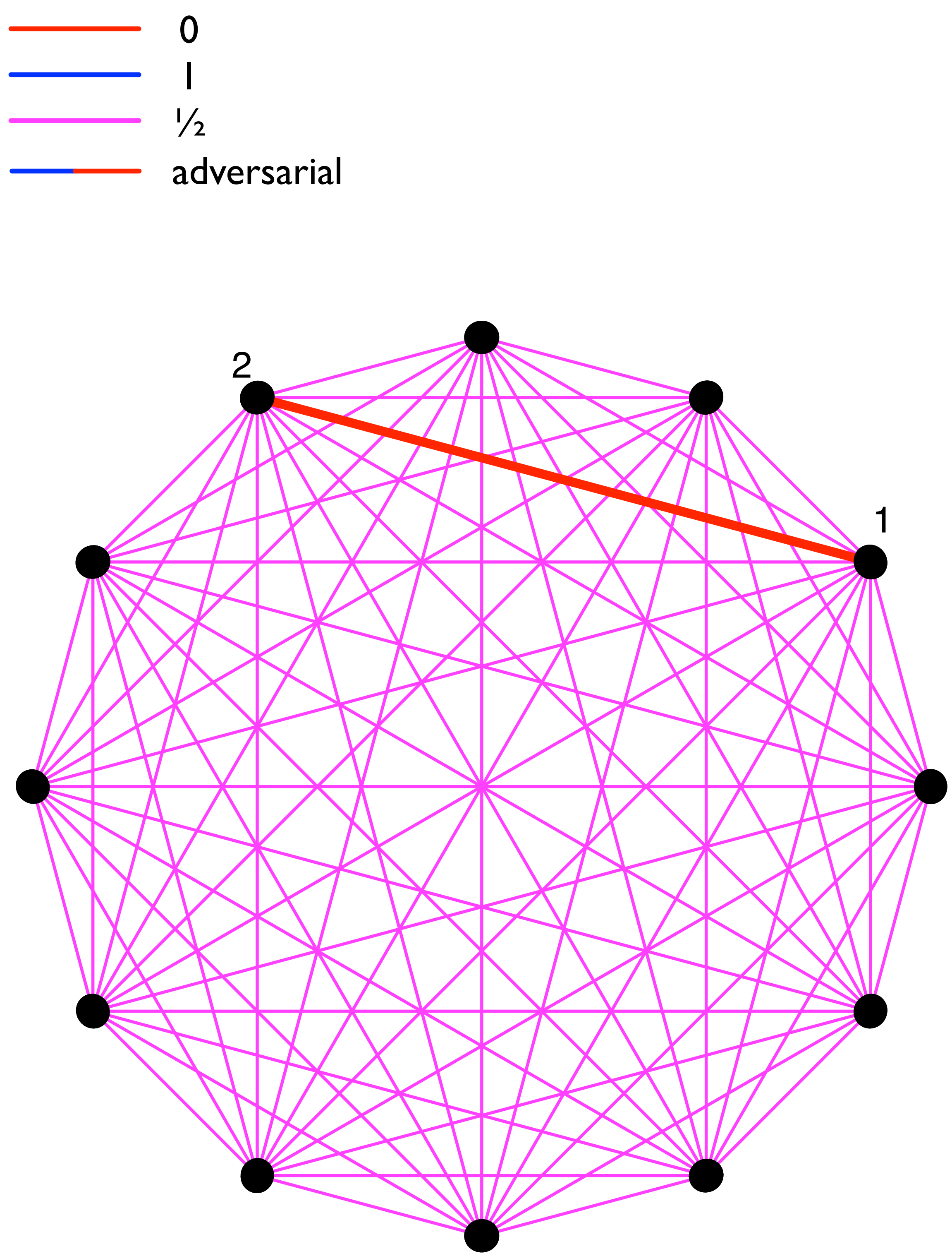


model graph for $k = 4$



- No homogenous sets of size k (independent of edges {2i-1,2i})
- For x≤ 2i-2 the exactly one between edges {x,2i-1} and {x,2i} is in the graph



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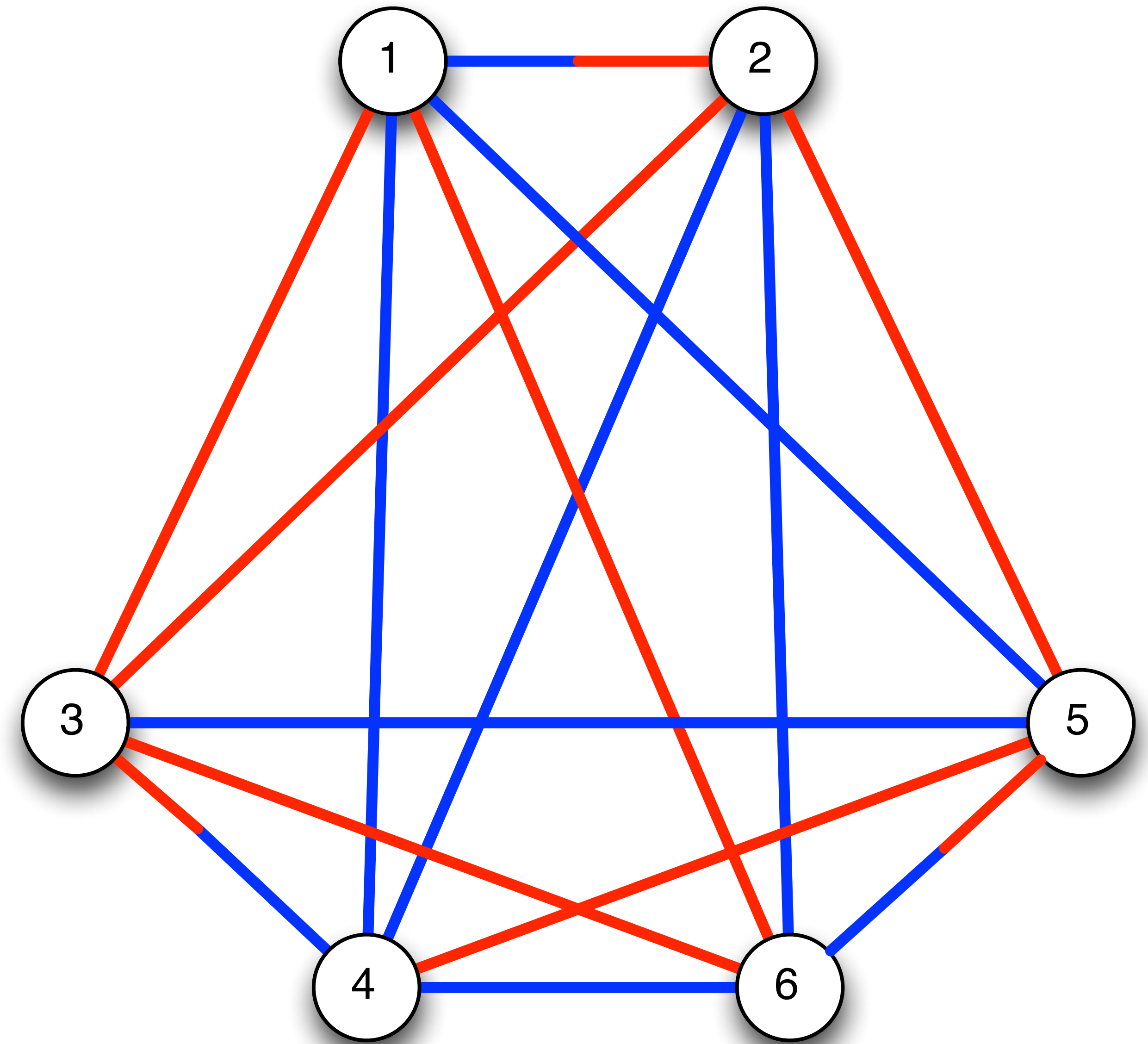

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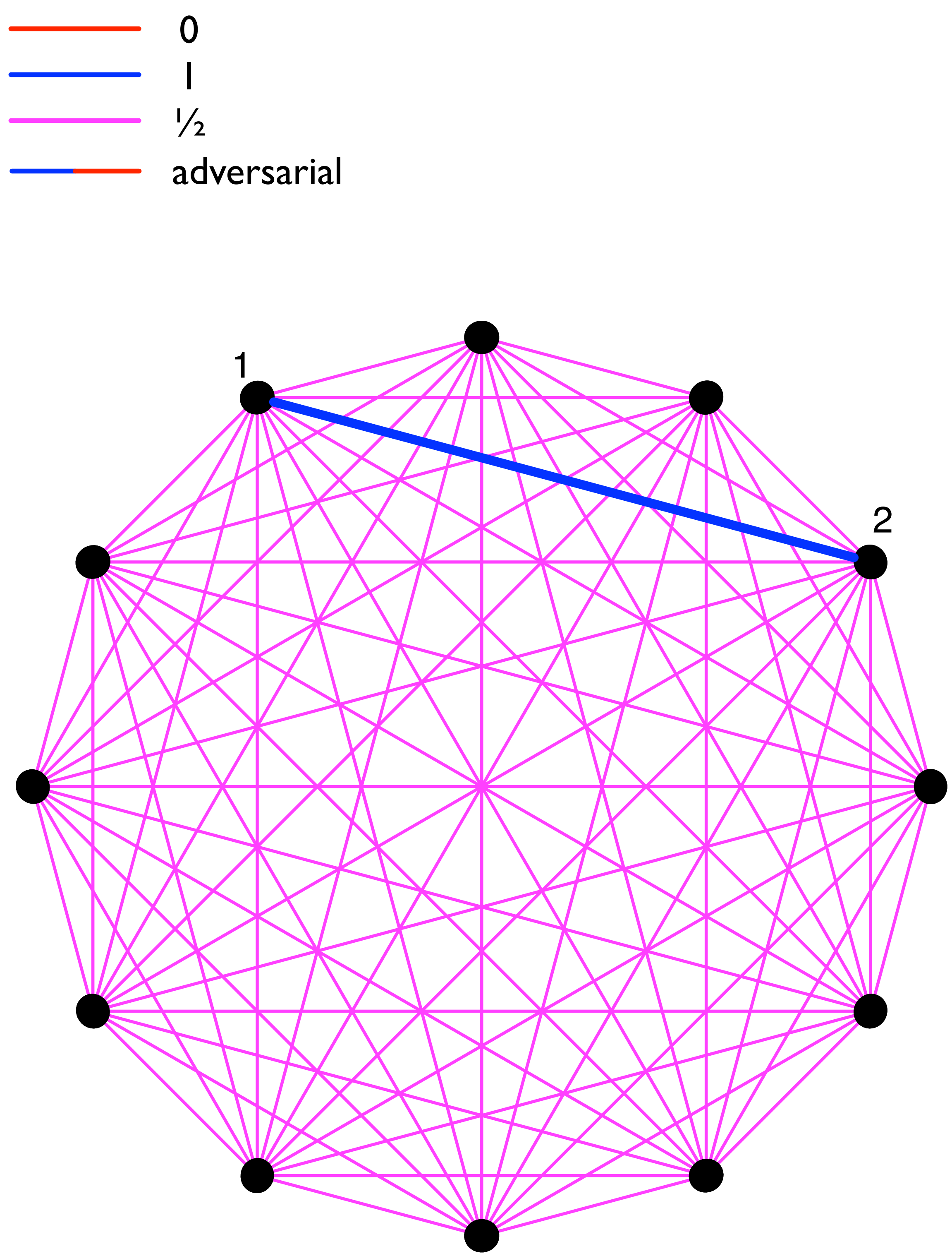

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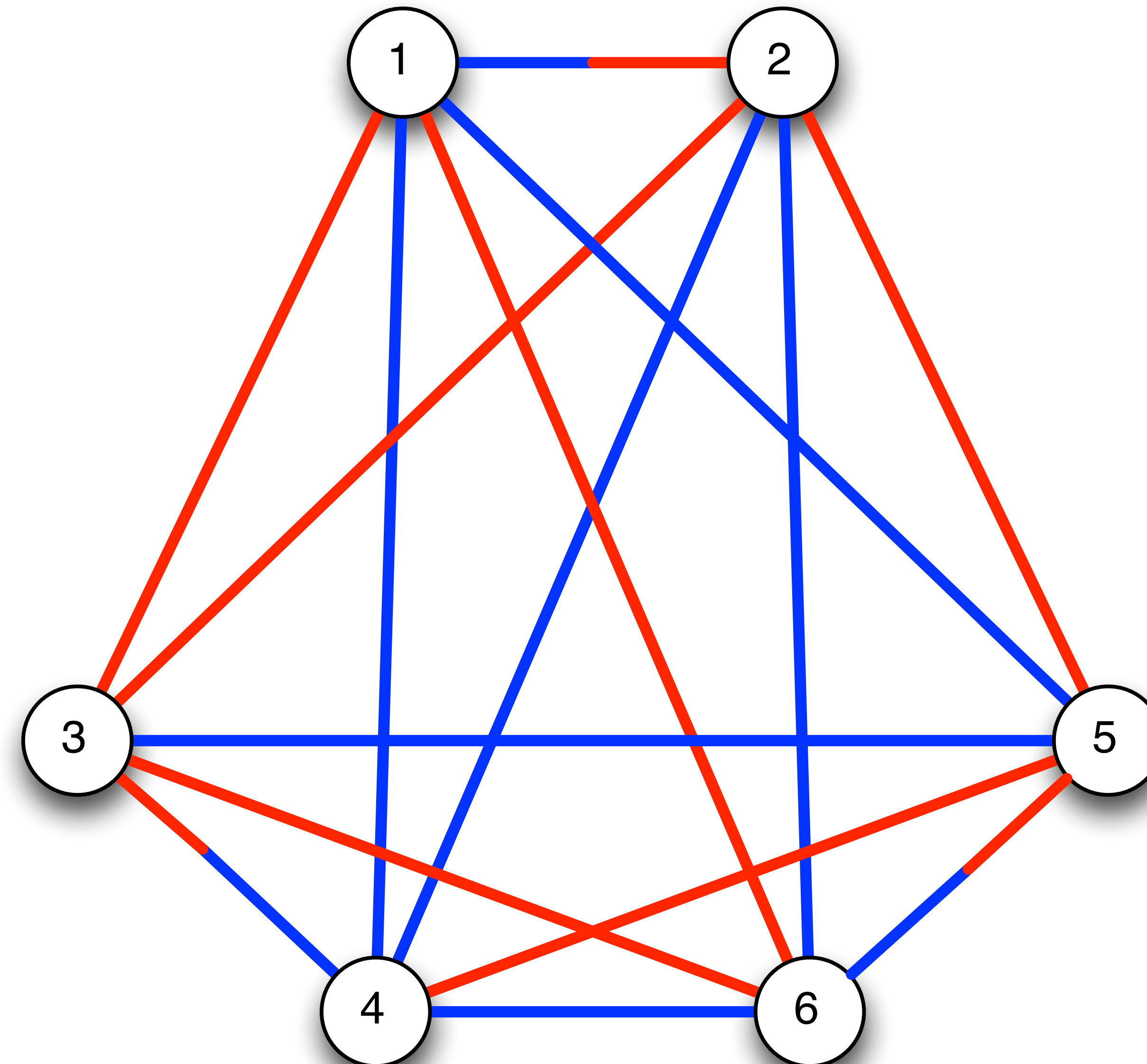

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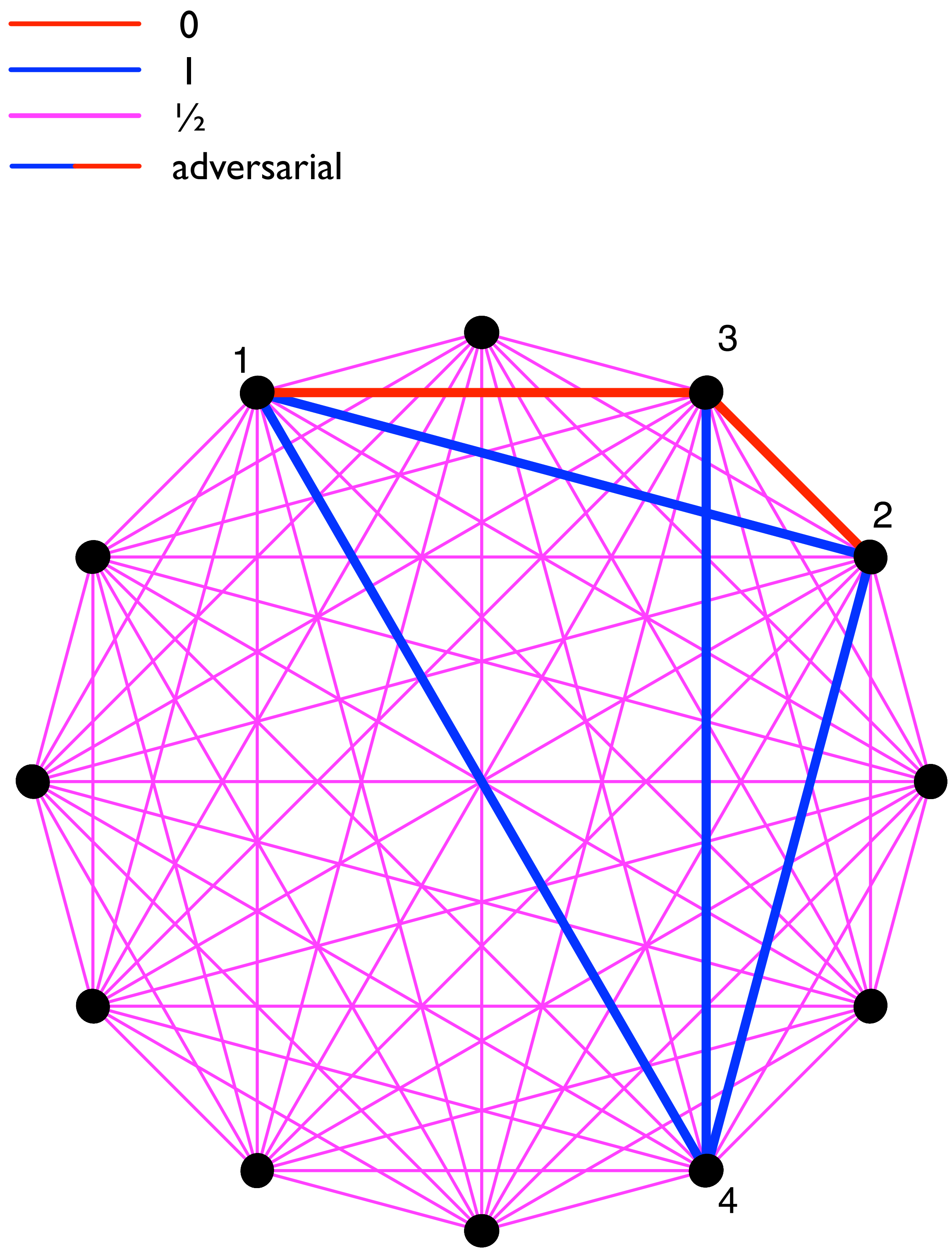

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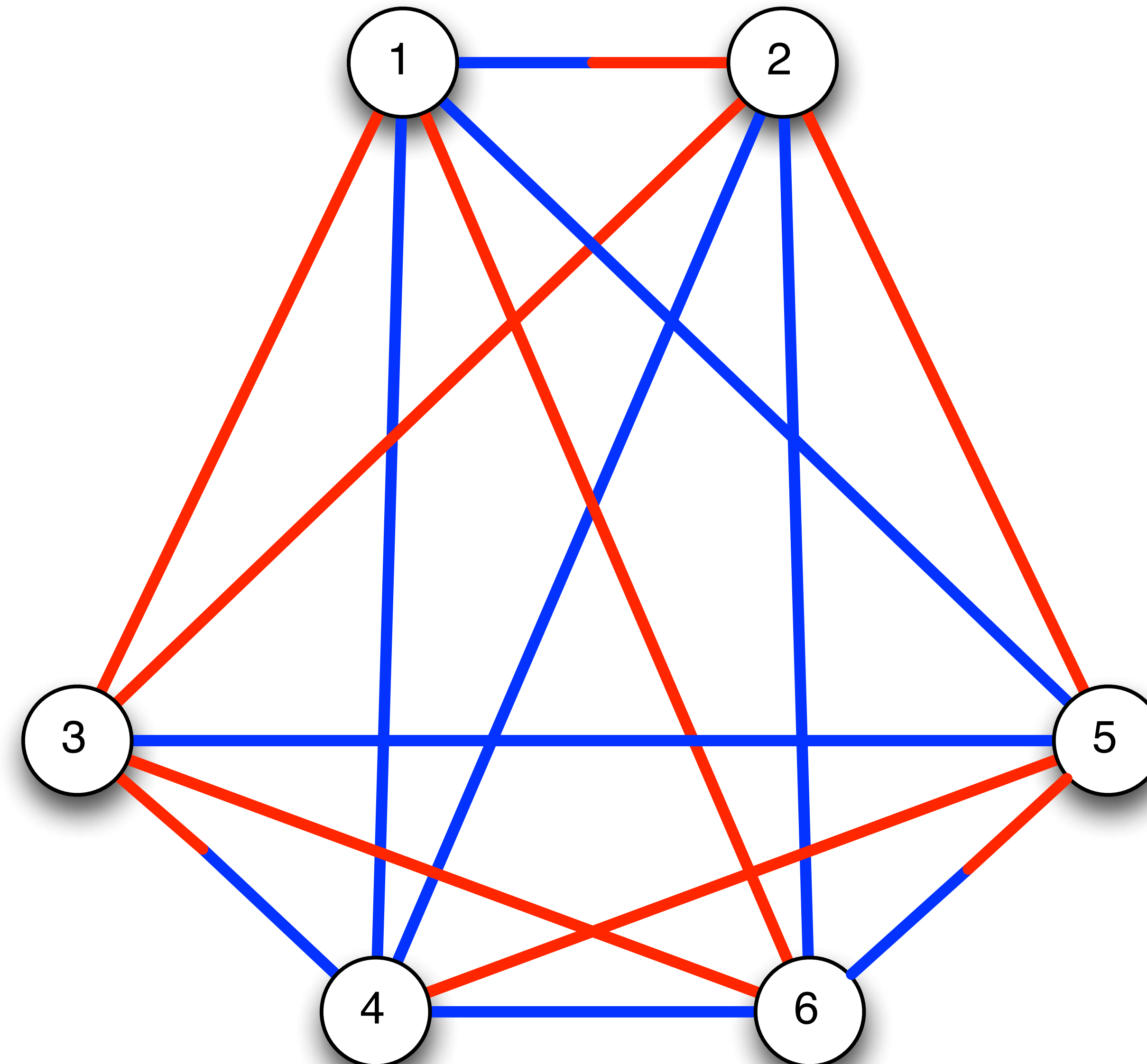

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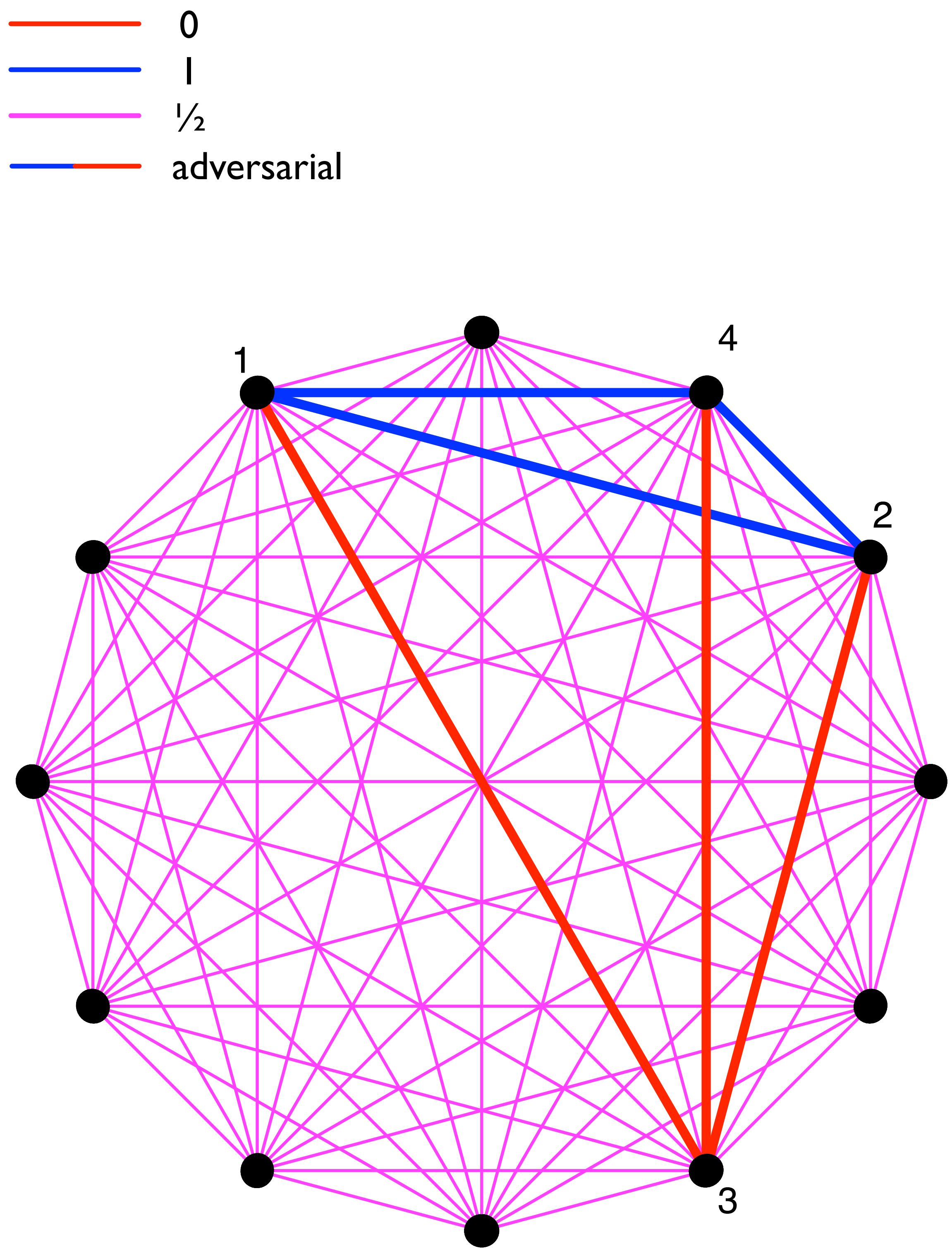

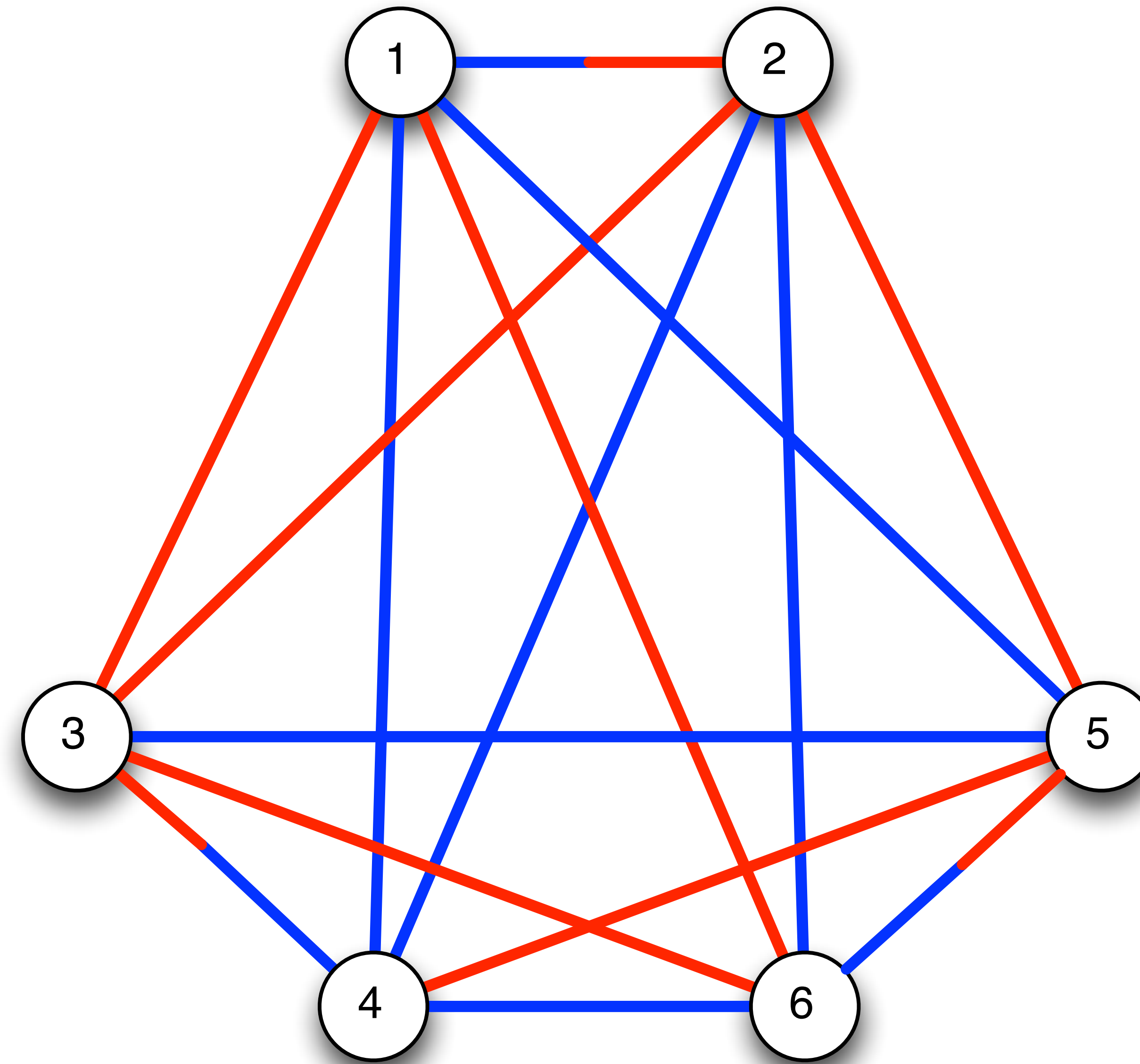

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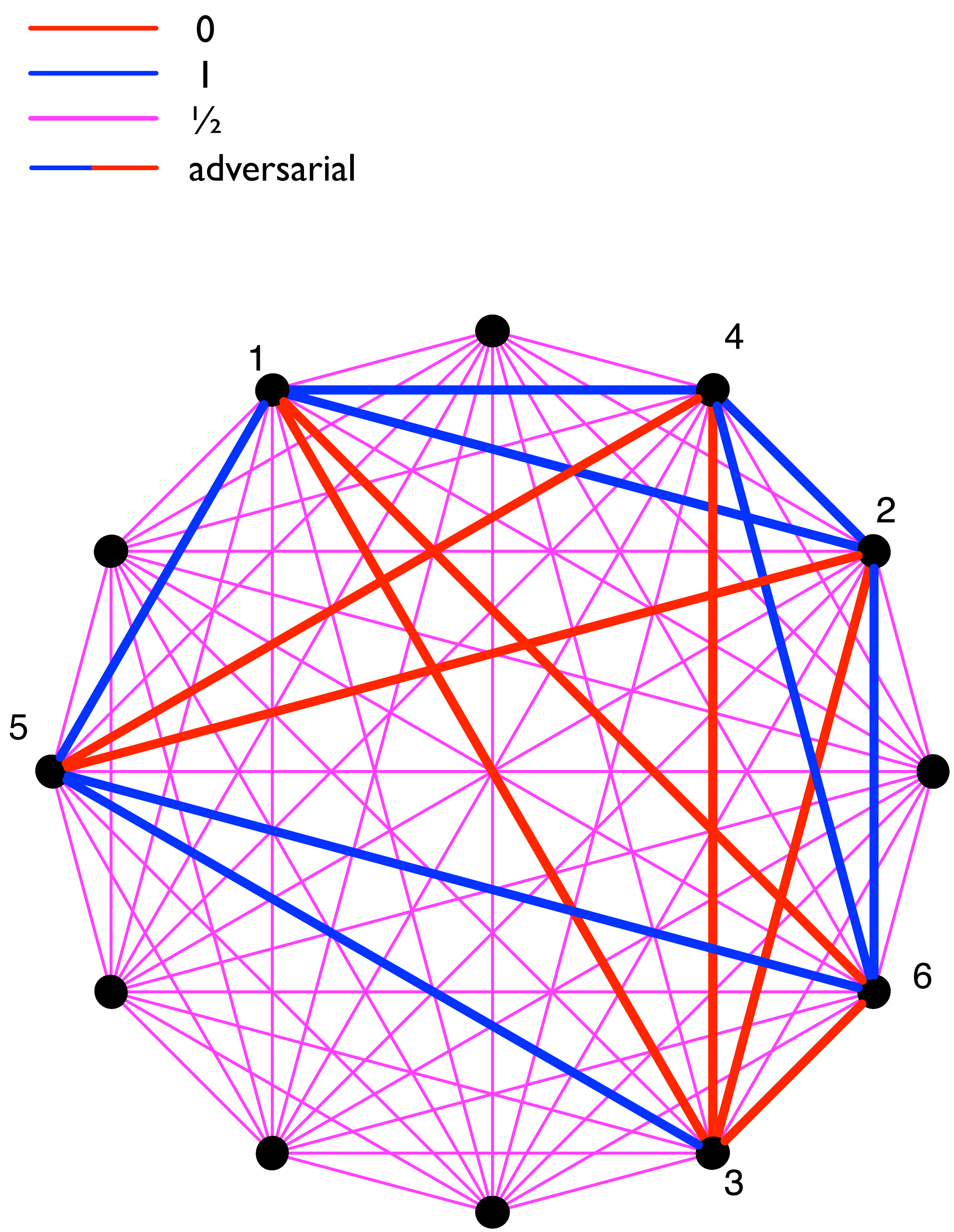

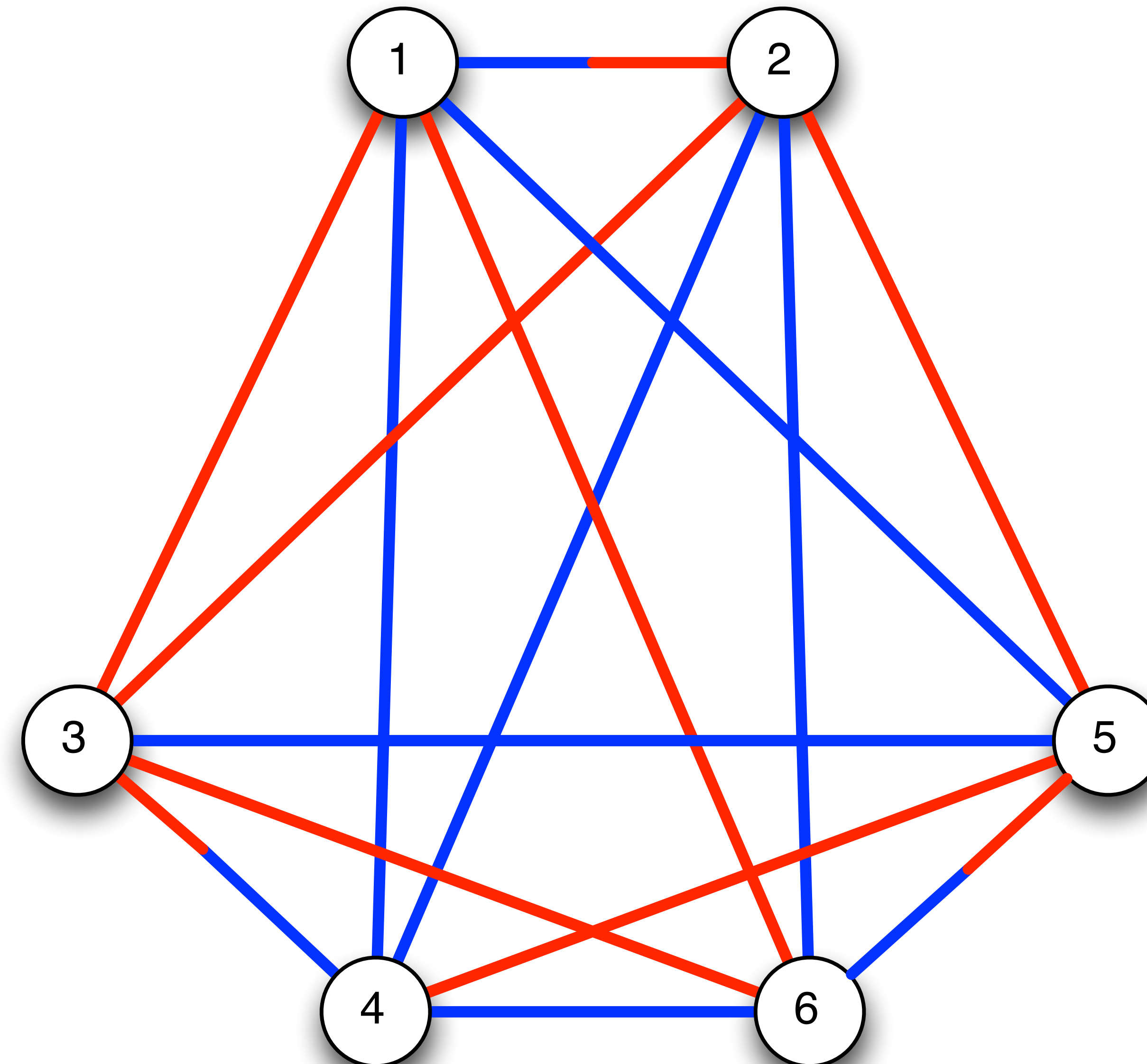

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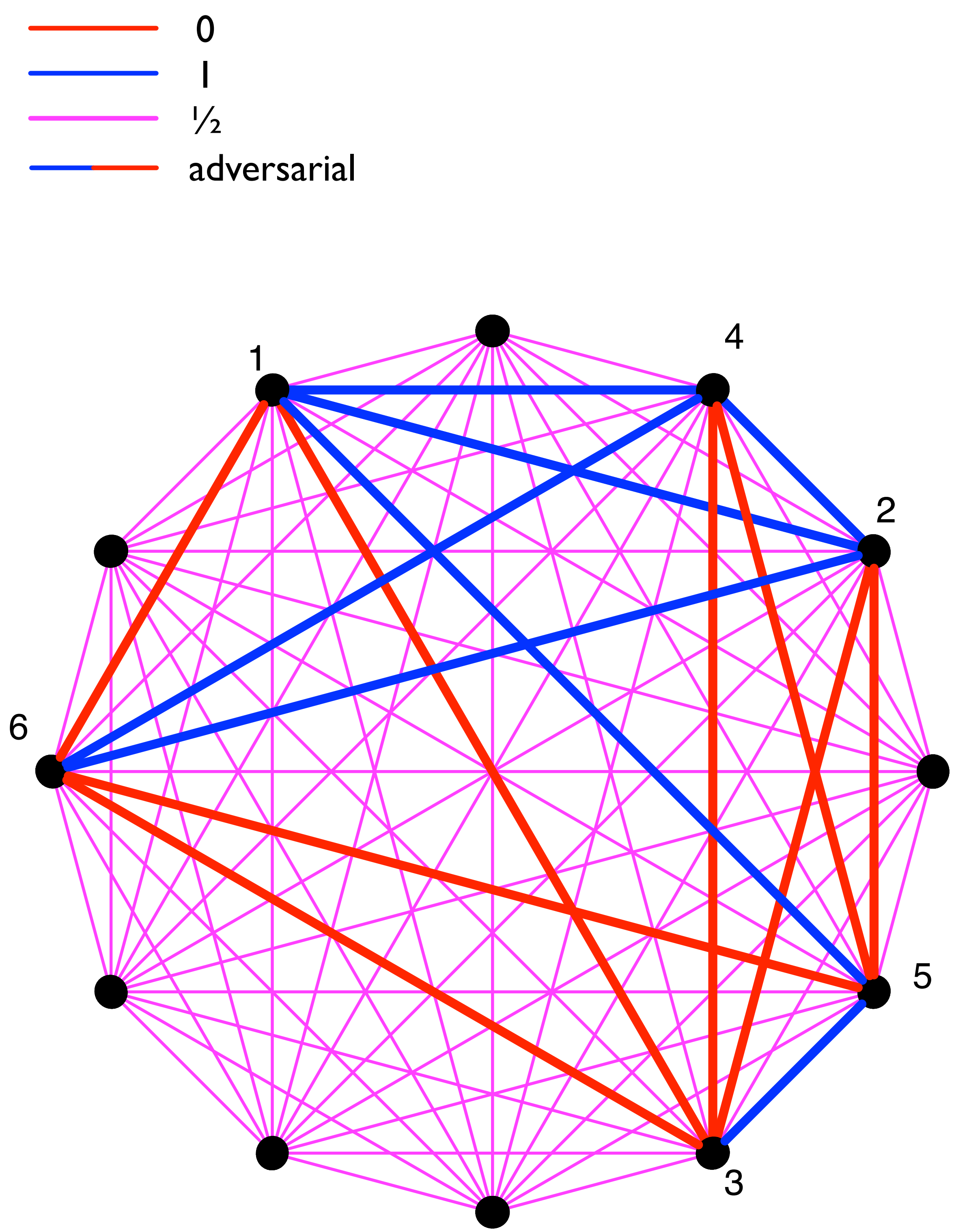

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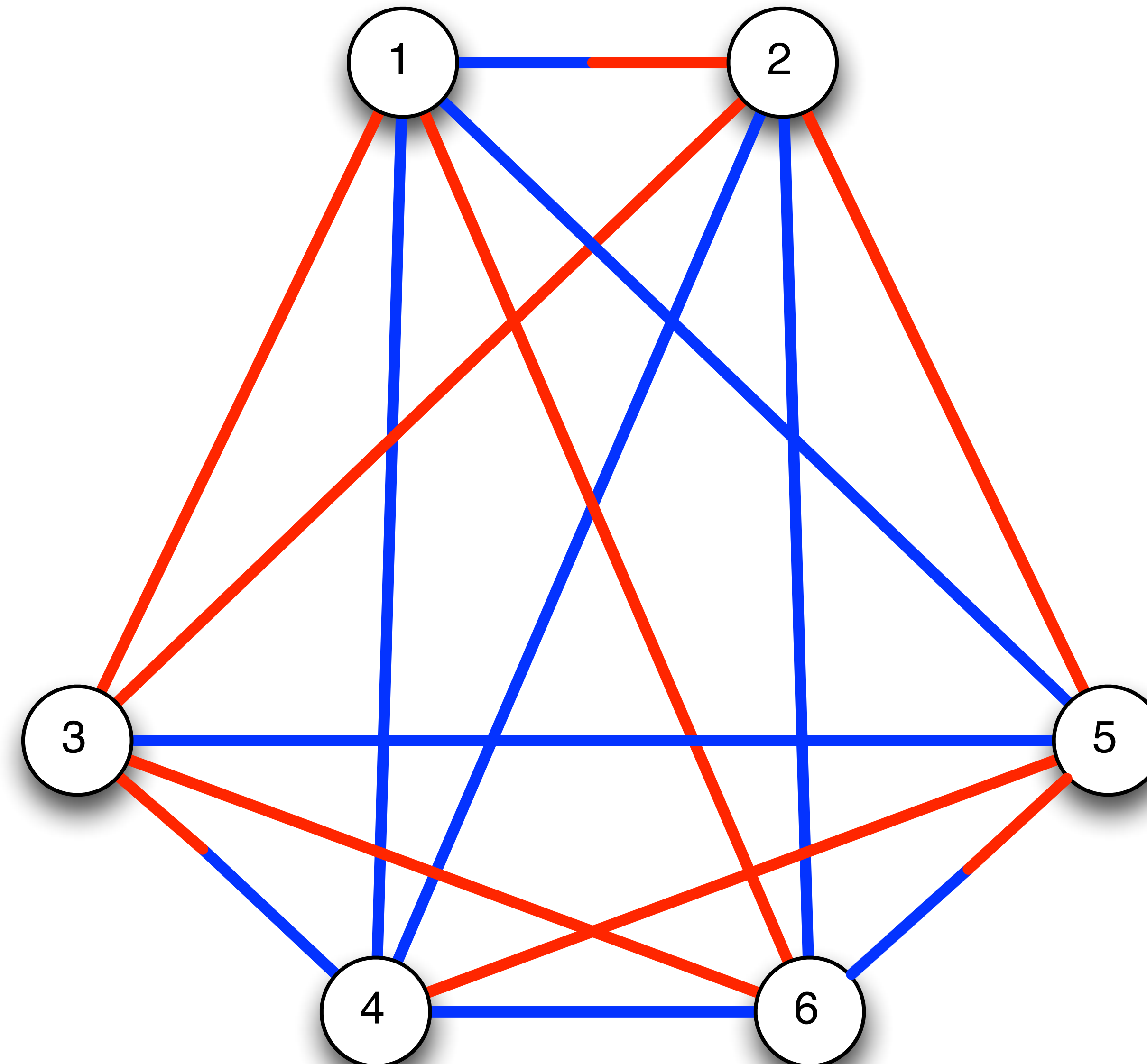

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By probabilistic method we can show that there is model graph of size

$$2^{k/2}$$

which gives a strategy for $2^{k/2-1}$ rounds.

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↓

Thm: our formula requires cutting planes refutations of rank $2^{k/2-1}$.

Summary

- Ramsey numbers R(k)
- proof system for Integer Programming
- upper bounding R(k) is "hard" for cutting planes
- a protection lemma for graph formulas.

Open problems

- New CP size lower bounds?
- Verifying witnesses for $R(k) > n$?

  (see [**L.**, Pudlák, Rödl, Thapen, 2013])

Thank you

questions?
remarks?
...
counterexamples? (© Jan Krajíček)