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INTEGRAL FLOW WITH DISJOINT BUNDLES

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Abstract. It is known by Sahni [2] that the integral flow problem with overlapped bundles is NP-complete. We show that this problem remains NP-complete even if the bundles are disjoint.

1. Definition of the problem

Let $D = (N, E)$ be a directed graph with specified nodes $s, t \in N$. A mapping $f : E \to \mathbb{N}_0$ is called an integral flow, if for each node $v \in N \setminus \{s, t\}$,

$$\sum_{(u,v) \in E} f((u,v)) = \sum_{(v,u) \in E} f((v,u)).$$

The flow value $v(f)$ is defined by:

$$v(f) = \sum_{(u,t) \in E} f((u,t)) - \sum_{(t,u) \in E} f((t,u)).$$

The integral flow problem with bundles is specified as follows:

Problem: Integral flow with bundles.

Instance: Directed graph $D = (N, E)$, specified nodes $s, t \in N$, sets $I_1, \ldots, I_k \subset E$, capacities $c_1, \ldots, c_k \in \mathbb{N}$, and $R \in \mathbb{N}$.

Question: Is there an integral flow $f : E \to \mathbb{N}_0$ with flow value $v(f) \geq R$ such that $\sum_{e \in I_j} f(e) \leq c_j$ for each $j \in \{1, \ldots, k\}$?

2. Complexity result

In an earlier paper Sahni has shown that this problem is NP-complete, if the sets $I_1, \ldots, I_k$, also called bundles, are overlapped. Sahni has given a transformation from the independent set problem to the integral flow problem with bundles. For each undirected graph $G = (N, E)$ he has constructed a directed graph $D = (N \cup \{s, t\}, \{(s, n), (n, t) | n \in N\})$ with bundles:

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(1) $I_e = \{(n_1, t), (n_2, t)\}$ for each edge $e = (n_1, n_2) \in E$,

(2) $I_{(s,n)} = \{(s, n)\}$ for each node $n \in N$,

(3) $I_{(n,t)} = \{(n, t)\}$ for each node $n \in N$ with degree $d(n) = 0$.

Each bundle is assigned the capacity 1. There is an integral flow of D with value $v(f) = R$, if and only if there is an independent set in G of cardinality R. We see, that the bundles are not disjoint, if there is at least one node $v \in N$ with degree $d(v) > 1$. For the subproblem with disjoint bundles we have the following complexity result:

THEOREM 1. *The integral flow problem remains NP-complete, if the bundles are disjoint.*

PROOF. Clearly, the flow problem is in NP. We give a transformation from the satisfiability problem, which is shown to be NP-complete [1], to the integral flow problem.

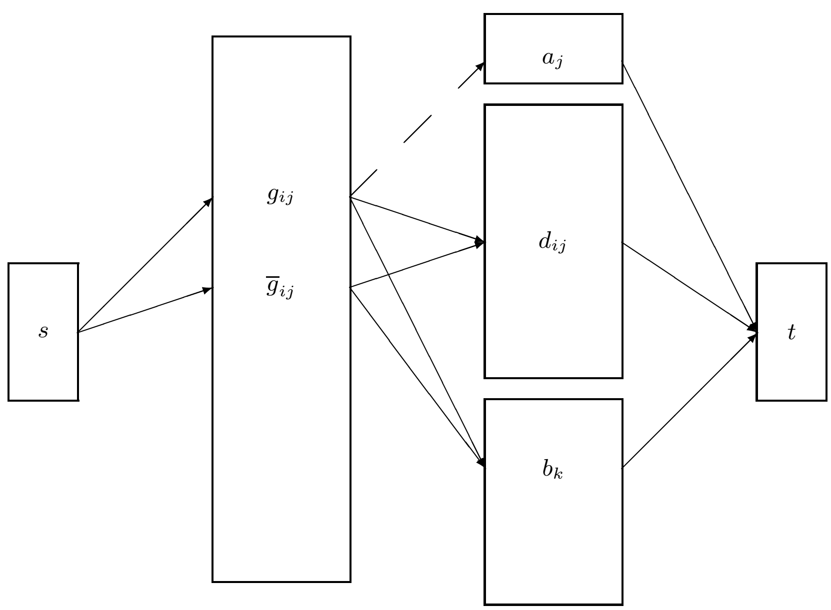

Fig. 1: The complete graph construction

Let $\alpha = (c_1 \wedge \ldots \wedge c_m)$ be a formula in conjunctive normal form with clauses $c_i = (y_{i1} \vee y_{i2} \vee y_{i3})$ and variables $V = \{x_1, \ldots, x_n\}$. We construct a directed graph $D = (N, E)$ and a set of disjoint bundles with corresponding capacities 1. We define:

$$\begin{aligned} N \quad = \quad & \{g_{ij}, \overline{g}_{ij}, d_{ij} | 1 \le i \le n, 1 \le j \le m\} \quad \cup \\ & \{b_k | 1 \le k \le m(n-1)\} \quad \cup \\ & \{a_j | 1 \le j \le m\} \cup \{s, t\} \end{aligned}$$

$$
\begin{array}{rcll}
E & = & \{(s,g_{ij}),(s,\overline{g}_{ij}),(g_{ij},d_{ij}),(\overline{g}_{ij},d_{ij})|1 \le i \le n, 1 \le j \le m\} & \cup \\
& & \{(d_{ij},t),(a_j,t)|1 \le i \le n, 1 \le j \le m\} & \cup \\
& & \{(g_{ij},b_k),(\overline{g}_{ij},b_k)|1 \le i \le n, 1 \le j \le m, 1 \le k \le m(n-1)\} & \cup \\
& & \{(g_{ij},a_j)|x_i \in \{y_{j1},y_{j2},y_{j3}\}, 1 \le i \le n, 1 \le j \le m\} & \cup \\
& & \{(\overline{g}_{ij},a_j)|\overline{x}_i \in \{y_{j1},y_{j2},y_{j3}\}, 1 \le i \le n, 1 \le j \le m\} & \cup \\
& & \{(b_k,t)|1 \le k \le m(n-1)\} &
\end{array}
$$

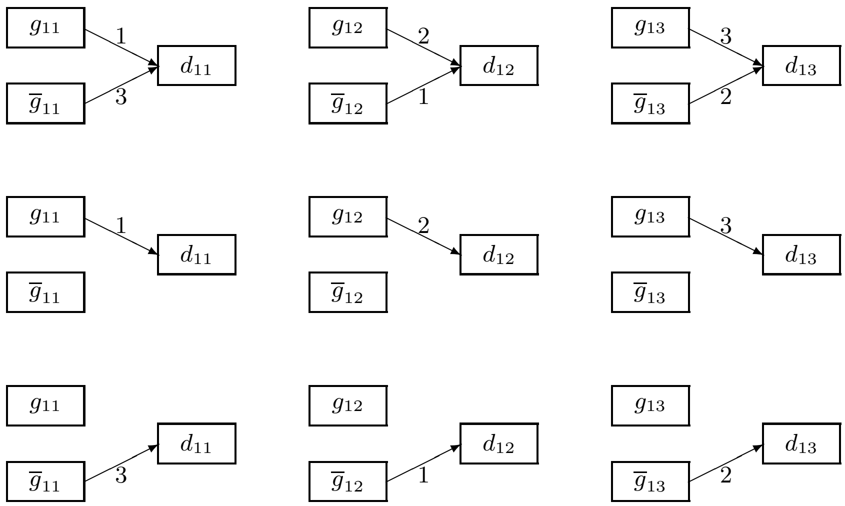

Fig. 2: The variable setting

The graph construction is illustrated in Figure 1. As bundles we have for each $1 \le i \le n$, $1 \le j \le m$:

$$I_{(i-1)m+j} = \{(g_{ij}, d_{ij}), (\overline{g}_{i(j \,mod\, m+1)}, d_{i(j \,mod\, m+1)})\},$$

and for each other edge $e \in E$ we take a bundle $I_e = \{e\}$.

We now claim: there is an integral flow of value $2{\cdot}n{\cdot}m$, if and only if the formula α is satisfiable.

First note that we must have a flow of size one through the edges (s, g_{ij}), $(s, \overline{g}_{ij})$, (d_{ij}, t), (a_j, t) and (b_j, t). The idea of the variable setting is given in Figure 2. Using the bundles (the indices of the bundles are drawn as edge labels in the Figure), there exist only two feasible choices for the flow between nodes $g_{1j}, \overline{g}_{1j}$ and d_{1j} for $j = 1, 2, 3$.

Suppose α is true. Let $\psi : \{x_1, \ldots, x_n\} \to \{0, 1\}$ be an assignment of values to the variables with $\psi(c_j) = 1$ for all $1 \le j \le m$. Then, we define as flow $f((g_{ij}, d_{ij})) = 1 - \psi(x_i)$ and $f((\overline{g}_{ij}, d_{ij})) = \psi(x_i)$ for $1 \le i \le n$, $1 \le j \le m$. Since $\psi(c_j) = 1$ for each $1 \le j \le m$, we can define $k_j \in \{1, 2, 3\}$ as the smallest index with $\psi(y_{jk_j}) = 1$. If the literal y_{jk_j} is equal to a non-complemented variable x_i, then we take $f((g_{ik_j}, a_j)) = 1$. If y_{jk_j} is equal to a complemented variable $\overline{x}_i$, then we take $f((\overline{g}_{ik_j}, a_j)) = 1$. The other flow, which arrives at g_{ij} or $\overline{g}_{ij}$ and goes not through a vertex d_{ij} or a_j, we send through one of vertices b_k. In total, we get a feasible integer flow of size $2 \cdot n \cdot m$ with $\sum_{e \in I_j} f(e) \leq 1$ for each bundle I_j.

Suppose we have an integer flow of size $2 \cdot n \cdot m$ with $\sum_{e \in I_j} f(e) \leq 1$. Then, for each $1 \leq i \leq n$ either $f((g_{ij}, d_{ij})) = 1$ for $1 \leq j \leq m$ or $f((\overline{g}_{ij}, d_{ij})) = 1$ for $1 \leq j \leq m$. Using this fact, we define an assignment ψ of the variables as follows:

$$\psi(x_i) = f((\overline{g}_{ij}, d_{ij})).$$

Since $f((a_j, t)) = 1$ for every $1 \leq j \leq m$, exactly one of three edges which arrives at a_j has flow value one. This implies directly that $\psi(c_j) = 1$ for each $1 \leq j \leq m$ and that $\psi(\alpha) = 1$. $\square$

From the proof of the theorem we obtain also:

COROLLARY 1. *The integral flow problem with bundles remains NP - complete, if*

(1) for each pair $i, j \in \{1, \ldots, k\}$ with $i \neq j$, $I_i \cap I_j = \emptyset$,

(2) for each $j \in \{1, \ldots, k\}$, $|I_j| \leq 2$ and $c_j = 1$,

(3) for each $j \in \{1, \ldots, k\}$, if $(x, y), (x, y') \in I_j$, then $y = y'$,

(4) for each $j \in \{1, \ldots, k\}$, if $(x, y), (x', y) \in I_j$, then $x = x'$.

References

[1] M.R. GAREY and D.S. JOHNSON, *Computers and Intractability*, A guide to the theory of NP-completness, (Freeman, New York, 1979).

[2] S. SAHNI, *Computationally related problems*, SIAM Journal on Computing, 3 (1974), pp. 262-279.