Nordic Journal of Computing 1(1994), 268–272.

A NOTE ON HIGHER ORDER VORONOI DIAGRAMS

JOHN W. KRUSSEL
Lewis & Clark College
Mathematical Sciences
Portland, OR 97219, U.S.A.
krussel@lclark.edu

BARRY F. SCHAUDT
Lewis & Clark College
Mathematical Sciences
Portland, OR 97219, U.S.A.
schaudt@lclark.edu

Abstract. In this note we prove some facts about the number of segments of a bisector of two sites that are used in a higher order Voronoi diagram.

CR Classification: F.2.1

Key words: computational geometry, Voronoi diagrams

Let S be a set of n sites in the Euclidean plane. Informally, the Voronoi diagram is a subdivision of the plane into regions such that each point of a region has the same closest site. Aurenhammer [3] discusses the importance of Voronoi diagrams to computer scientists and presents a survey of the Voronoi diagram and its variants, including mathematical properties and algorithms. The variant that we are interested in here is the k^{th} order Voronoi diagram, denoted by $V_k(S)$. The k^{th} order Voronoi diagram is a partition of the plane into regions such that points in each region have the same k closest sites. There are many deterministic algorithms to compute $V_k(S)$ [7, 5, 4, 2]. Lee, Chazelle and Edelsbrunner [7, 4] describe many properties of $V_k(S)$. In this paper we are not concerned with algorithms to compute $V_k(S)$, instead, we will prove some properties about the number of segments of a bisector of two sites that are used in $V_k(S)$.

Most algorithms for Voronoi diagrams and variants of Voronoi diagrams assume non-degeneracy: no four sites are cocircular. For our purposes, this also implies no three sites lie on the same line. We will first make this assumption and then state what happens when we have degeneracies.

In order to prove some facts about the k^{th} order Voronoi diagram, we will transform the diagram to a three dimensional arrangement of planes using the standard lifting projection [4, 6]. That is, a site p_i with coordinates (x_1, x_2) is mapped to the plane H_i tangent to the paraboloid $x_3 = x_1^2 + x_2^2$ at the point $(x_1, x_2, x_1^2 + x_2^2)$. The arrangement of these planes is denoted by $A(H)$.

There is a one-to-one correspondence between the edges in $V_k(S)$ and the edges e in $A(H)$ with $k-1$ planes above e. Corresponding to the perpen-

Received December 1993. Accepted August 1994.

dicular bisector of the segment $\overline{p_i p_j}$, where $p_i, p_j \in S$, is the intersection l_{ij} between H_i and H_j. The other planes break up l_{ij} into segments, two of these segments are rays. We will soon need the following simple lemma.

LEMMA 1. *With the non-degeneracy assumption, each plane H_m, with $m \neq i$ and $m \neq j$, intersects l_{ij}.*

PROOF. The non-degeneracy assumption implies that for each $p_m \in S$, the circumsphere of p_i, p_j and p_m exists. The center of the circumsphere is a Voronoi vertex in $V_k(S)$ for some k. Thus, the perpendicular bisector of segment $\overline{p_i p_m}$ intersects the perpendicular bisector of $\overline{p_i p_j}$ for each $p_m \in S$. Therefore, each plane H_m intersects l_{ij}. □

We now define a directed graph G that we will use to prove our facts about the number of segments of a bisector in a k^{th} order Voronoi diagram. Each vertex in G has a value and G is determined by two parameters, the value of the source s and n. An example of such a graph G with source value $r-1$ is shown in Figure 1. The graph has vertices on a rectangular lattice that has $s+1$ columns and $n-1-s$ rows. The source is the upper left vertex of the lattice. Each internal vertex v of G has outdegree 2, a right edge and a down edge. The right edge leads to a vertex with value $v-1$ and the down edge leads to a vertex of value $v+1$. Vertices on the boundary have either a right edge or a down edge similarly defined. The lower right vertex, called the sink, has outdegree 0 and has value $n-2-s$. We will assume that the value of the source is less than or equal to the value of the sink. If not, we can construct a new graph by reversing the direction of the edges. The value of the source in this graph will be less than the value of the sink.

We now look at the relation of graph G to a traversal of line l_{ij} as we move along l_{ij} from $-\infty$ to ∞.

Suppose one ray of l_{ij} has $r-1$ planes above it and thus corresponds to an infinite edge in $V_r(S)$. Lemma 1 states that each of the other $n-2$ planes intersects l_{ij} and because of our non-degeneracy assumption, no two intersect at the same point. Along l_{ij} the number of planes above changes by exactly one at each intersection point. In particular, at the other infinite segment, each of the $r-1$ "above"-planes have changed into "below"-planes and vice versa. Hence, there are $n-2-(r-1) = n-r-1$ planes above this ray and so its projection appears in $V_{n-r}(S)$. It is now easy to see that traveling along l_{ij} corresponds to a path of length $n-2$ from the source to the sink of G. Using this graph, we can easily prove several results about the bisector segments used in a k^{th} order Voronoi diagram.

FACT 1. At most $\min(k, n-k)$ segments of any bisector can appear in $V_k(S)$.

PROOF. Without loss of generality, suppose $k \leq n-k$. Any bisector of sites p_i and p_j corresponds to the intersection l_{ij} of two planes. In the graph G defined above for l_{ij} the label $k-1$ appears at most k times. □

Similarly, at most k segments of a bisector can appear in $V_{n-k}(S)$. Using the graph G, we can obtain the following stronger result.

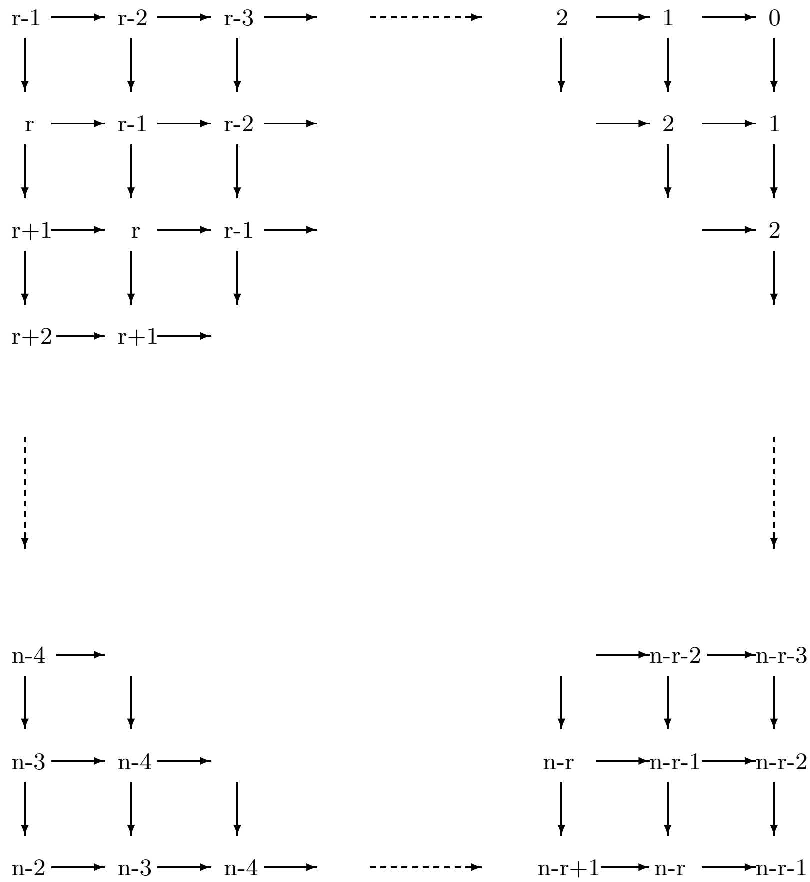

Fig. 1: The directed graph G with source value $r-1$.

FACT 2. The number of finite segments of a bisector appearing in either $V_k(S)$ or $V_{n-k}(S)$ cannot exceed $\min(k, n-k)$.

PROOF. Without loss of generality, suppose $k \le n-k$. The path of length $n-2$ in G from source to sink with source $r-1$ contains $n-1$ vertices. Since vertices labeled $k-1$ and those labeled $n-k-1$ are at a distance of at least $n-2k$, at least $n-2k-1$ vertices of the path are used getting from one to the other. Thus, there remain $n-1-(n-2k-1) = 2k$ unaccounted for in the path. Now since two vertices with the same label are at distance 2 and vertices whose label differing by more than one are at least distance 2, at most k of these vertices will have the label $k-1$ or $n-k-1$. □

FACT 3. If a ray of a bisector B appears in the k^{th} order Voronoi diagram, then at most $\min(k, k', n-k)$ segments of B appear in the $k'^{\,\text{th}}$order Voronoi diagram.

PROOF. Without loss of generality, suppose $k \le n-k$. Let B be the bisector, then B is associated with an intersection of two planes l_{ij} with graph G. Because a ray of B appears in the k^{th} order Voronoi diagram, the source of G is labeled $k-1$. Therefore, a label $k'-1$ appears at most $\min(k,k')$ times in G. □

As a direct result of this fact, any bisector with a ray in $V_1(S)$ will have exactly one segment in $V_k(S)$ for $1 \le k \le n-1$.

FACT 4. If a finite segment of a bisector B appears in $V_k(S)$ there will be at most $\min(k+1, n-k+1, k')$ segments of B in $V_{k'}(S)$ for $1 \le k' \le n-1$.

PROOF. Again, we will assume that $k \le n-k$. If $k \ge k'$ then from Fact 1, there are at most k' segments of a bisector in $V_{k'}(S)$.

Now suppose $k < k'$ and $r \le n-r$. Let G be the graph associated with the bisector and let it have source $r-1$. We now have 2 cases.

For the first case, $k < r$. Because a bisector segment appears in $V_k(S)$, a vertex v with label $k-1$ appears on the path in G from the source to the sink. We account for one vertex of each label $k-1, k, k+1, \ldots, n-r-1$ in the parts of the path going from the source to v and from v to the sink. Thus, $n-1-(n-2k+1) = 2k-2$ vertices are unaccounted for and, as in the proof of Fact 2, at most $k-1$ of these can have the same label k'. There is a vertex labeled $k'-1$ on the path from the source to v and another on the path from v to the sink. Therefore, at most $k+1$ vertices have label $k'-1$.

For the second case, $k \ge r$ and $r \le n-r$. The length of any diagonal in G (nodes with the same label) is at most r. Therefore, the number of segments of the bisector in $V_{k'}(S)$ is at most r, which does not exceed $\min(k+1, k')$.

If $r > n-r$, then we can apply the same argument by constructing the graph by traversing the line l_{ij} associated with the bisector in the opposite direction. The effect of traversing the line in the opposite direction is that the roles of the source and sink are reversed. □

From this fact and Fact 2, if a finite segment of a bisector B appears in $V_1(S)$ there will be at most 2 segments of B in $V_k(S)$ for $2 \le k \le n-2$ and no segment of B will appear in $V_{n-1}(S)$.

If we allow degeneracies then more than three sites are cocircular or more than two sites lie on the same line. In either case, we can still use the same graph G with a bisector. If more than three sites are cocircular, then some of the bisector segments will have length 0. If more than two sites lie on the same line then the perpendicular bisectors of these sites are parallel. In this case, a path in G from the source to the sink will be too long, because at least one of the planes will not intersect the line l_{ij} associated with G. Therefore, the facts hold even if we have degeneracies.

Aurenhammer [1] defines power diagrams and higher order power diagrams. Because the arrangement of planes for power diagrams is similar to the arrangement of planes for a Voronoi diagram, our facts also hold for k^{th} order power diagrams in the plane.

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