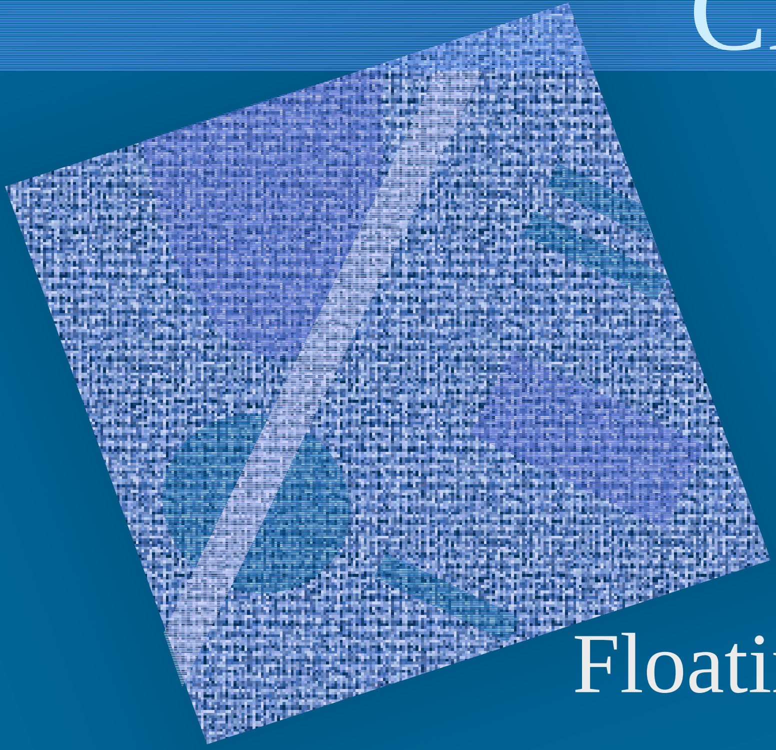
Computer Arithmetic
Ch 8



ALU

Integer Representation

Integer Arithmetic

Floating-Point Representation

Floating-Point Arithmetic

Arithmetic Logical Unit (ALU) (2)

(aritmeettis-looginen yksikkö)

- Does all "work" in CPU

Rest is management!

  - integer & floating point arithmetic's
  - copy values from one register to another
  - comparisons
  - left and right shifts
  - branch and jump address calculations
  - load/store address calculations
- Control signals from CPU control unit
  - what operation to perform and when

ALU Operations (5)

- Data from/to internal registers (latches)
  - input data may have been copied from normal registers, or it may have come from memory
  - output data may go to normal registers, or to memory

Fig. 8.1

- Wait for maximum gate delay
- Result is ready
- Result may (also) be in flags

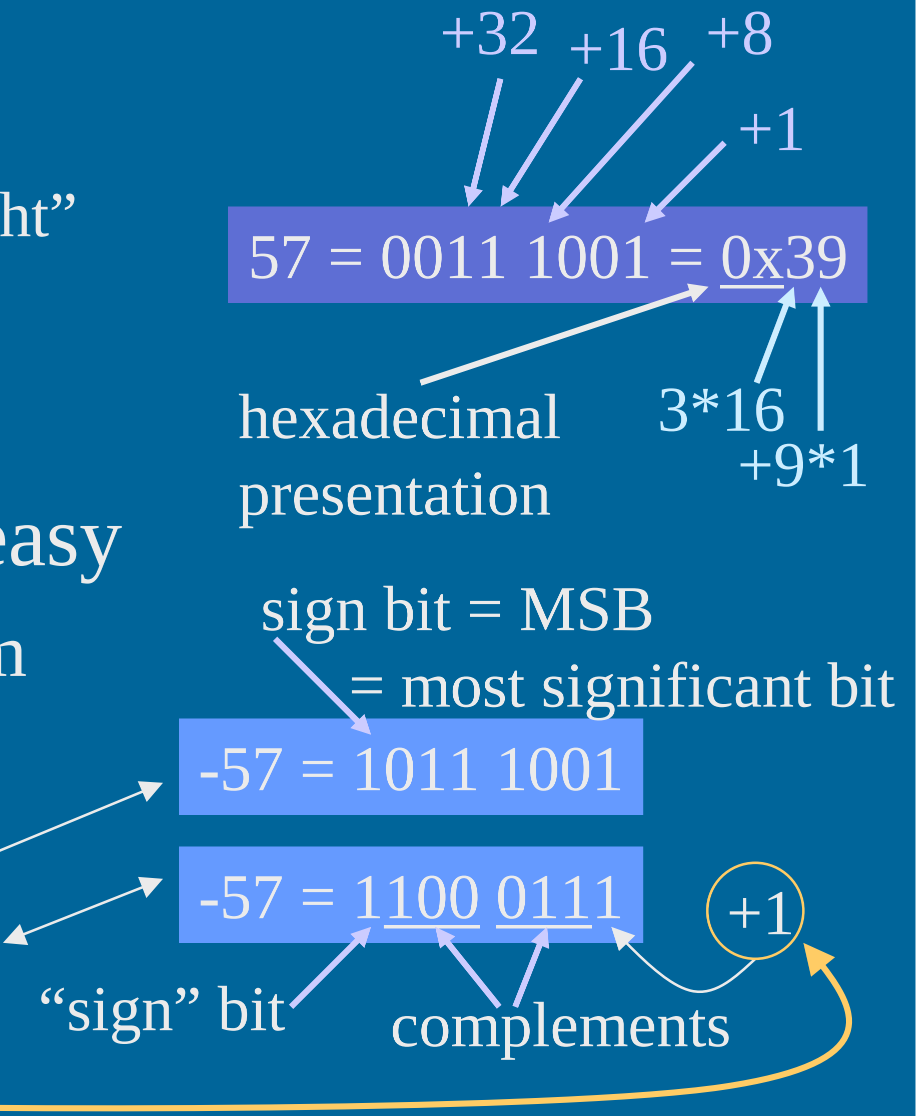
(lipuke)

- Flags may cause an interrupt

Integer Representation (8)

Everything with 0 and 1
    no plus/minus signs
    no decimal periods
        assumed "on the right"

- Unsigned integers
- Positive numbers easy
  - normal binary form
- Negative numbers
  - sign-magnitude
  - two's complement



57 = 0011 1001 = 0x39

-57 = 1011 1001

-57 = 1100 0111

Twos Complement

(kahden komplementti)

- Most used
- Have space for 8 bits?
  - use 7 bits for data and 1 bit for sign

+2 = 0000 0010
+1 = 0000 0001
0 = 0000 0000
-1 = 1111 1111
-2 = 1111 1110

  - just like in sign-magnitude or in one’s complement (but presentation is different)

ones complement: -0 = 1111 11$\underline{11}$

Why Two's Complement Presentation? (4)

- Math is easy to implement
  - subtraction becomes addition

$X-Y = X + (-Y)$

easy to do,
simple circuit

- Have just one zero
  - comparisons to zero easy
- Easy to expand to presentation with more bits
  - simple circuit

$57 = \underline{0}011\ 1001 = \underline{0000\ 0000}\ \underline{0}011\ 1001$

$-57 = \underline{1}100\ 0111 = \underline{1111\ 1111}\ \underline{1}100\ 0111$

sign extension

Why Two's Complement Presentation? (3)

- Range with n bits: $-2^{n-1} \ldots 2^{n-1} - 1$

$\ \ $8 bits: $-2^{7} \ldots 2^{7} - 1 = -128 \ldots 127$

32 bits: $-2^{31} \ldots 2^{31} - 1 = -2\,147\,483\,648 \ldots 2\,147\,483\,647$

- Overflow easy to recognise
  - add positive & negative: overflow not possible!
  - add 2 positive/negative numbers
    - if sign bit of result is different? $\Rightarrow$ overflow!

$$\begin{array}{r} 57 = 0011\ 1001 \\ +\ 80 = 0101\ 0000 \\ \hline 137 = \underline{1}000\ 1001 \end{array}$$

outside range

Why Two’s Complement Presentation? (5)

- Addition easy if one or both operands negative
  - treat them all as unsigned integers

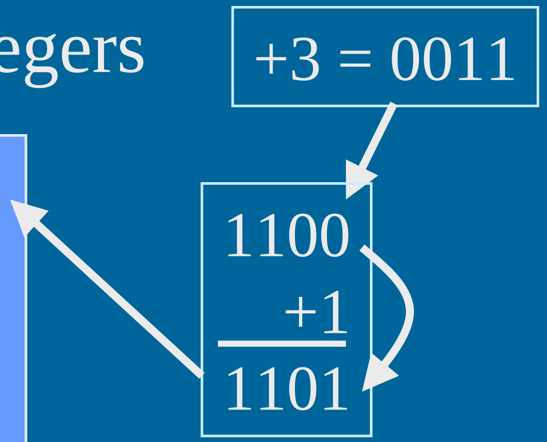
Same circuit works for both (except for overflow check)

13 = 1101
+1 = 0001
14 = 1110

Digits represent 4 bit unsigned numbers

-3 = 1101
+1 = 0001
-2 = 1110

Digits represent 4 bit two’s complement numbers



+3 = 0011

1100
+1
1101

Integer Arithmetic Operations

- Negation
- Addition
- Subtraction
- Multiplication
- Division

X = -Y

X = Y+Z

X = Y-Z

X = Y*Z

X = Y / Z

Integer Negation (6)

- Step 1: negate all bits
- Step 2: add 1
- **Step 3: special cases**
  - ignore carry bit
    - negate 0?
  - check that sign bit really changes
    - can not negate smallest negative
    - results in exception

57 = 0011 1001

1100 0110

+1

1100 0111

0 = 0000 0000

1111 1111

+1

-0 = 1 0000 0000

-128 = 1000 0000

bitwise not: 0111 1111

add 1: 1000 0000

Integer
Addition and Subtraction (4)

- Normal binary addition
  - 32 bit full adder?
- Ignore carry & monitor sign bit for overflow
- In case of SUB, complement 2nd operand
- 2 circuits
  - addition
  - complement

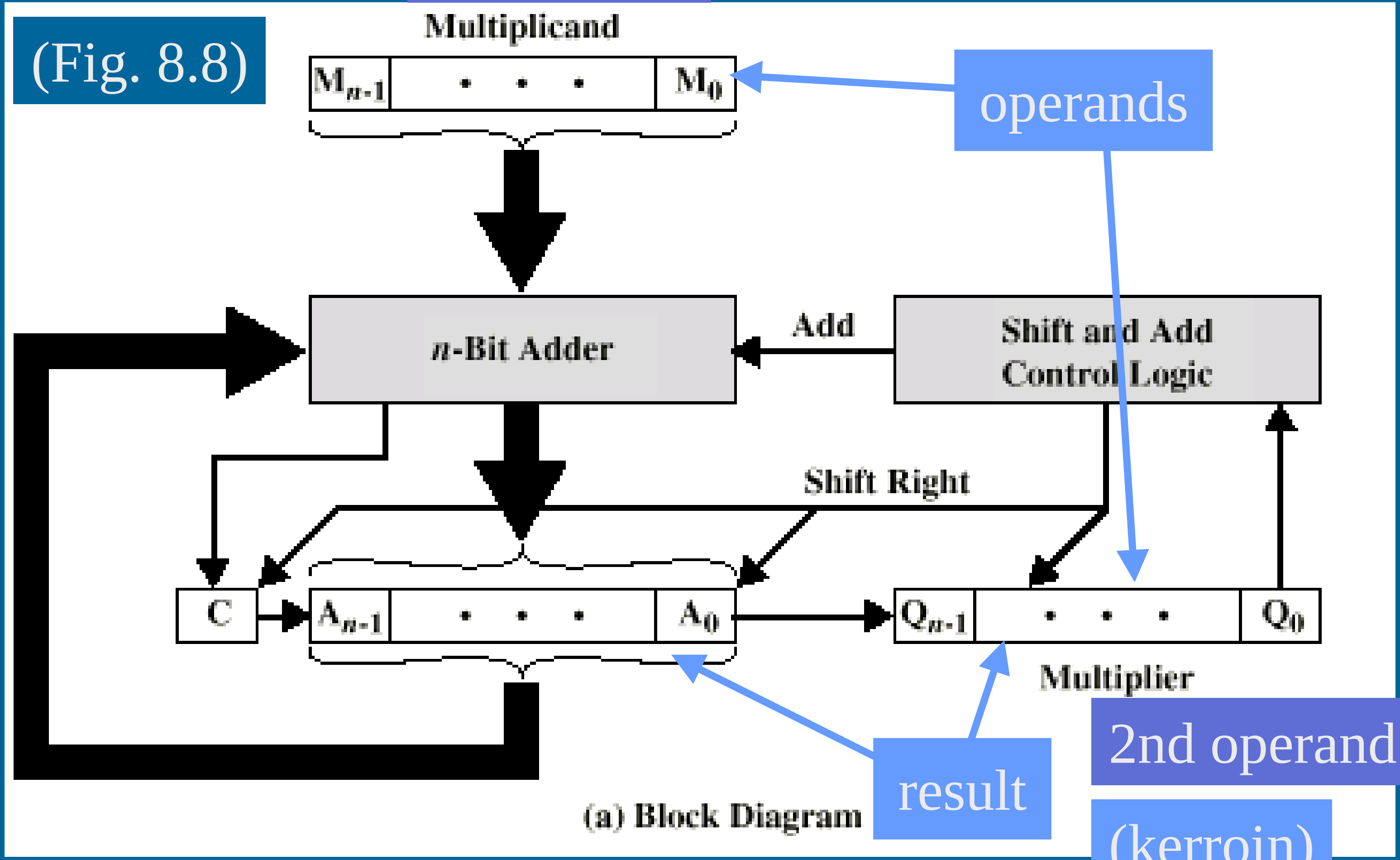
Fig. 8.6

Integer Multiplication (4)

- Complex
- Operands 32 bits ⇒ result 64 bits
- "Just like" you learned at school
  - optimised for binary data
    - it is easy to multiply with 0 or 1!
- Simpler case with unsigned numbers
  - simple circuits
    - adder
    - shifter
    - wires

Fig. 8.7

Unsigned Multiplication Example



1st operand

(kerrottava)

(Fig. 8.8)

operands

result

2nd operand

(kerroin)

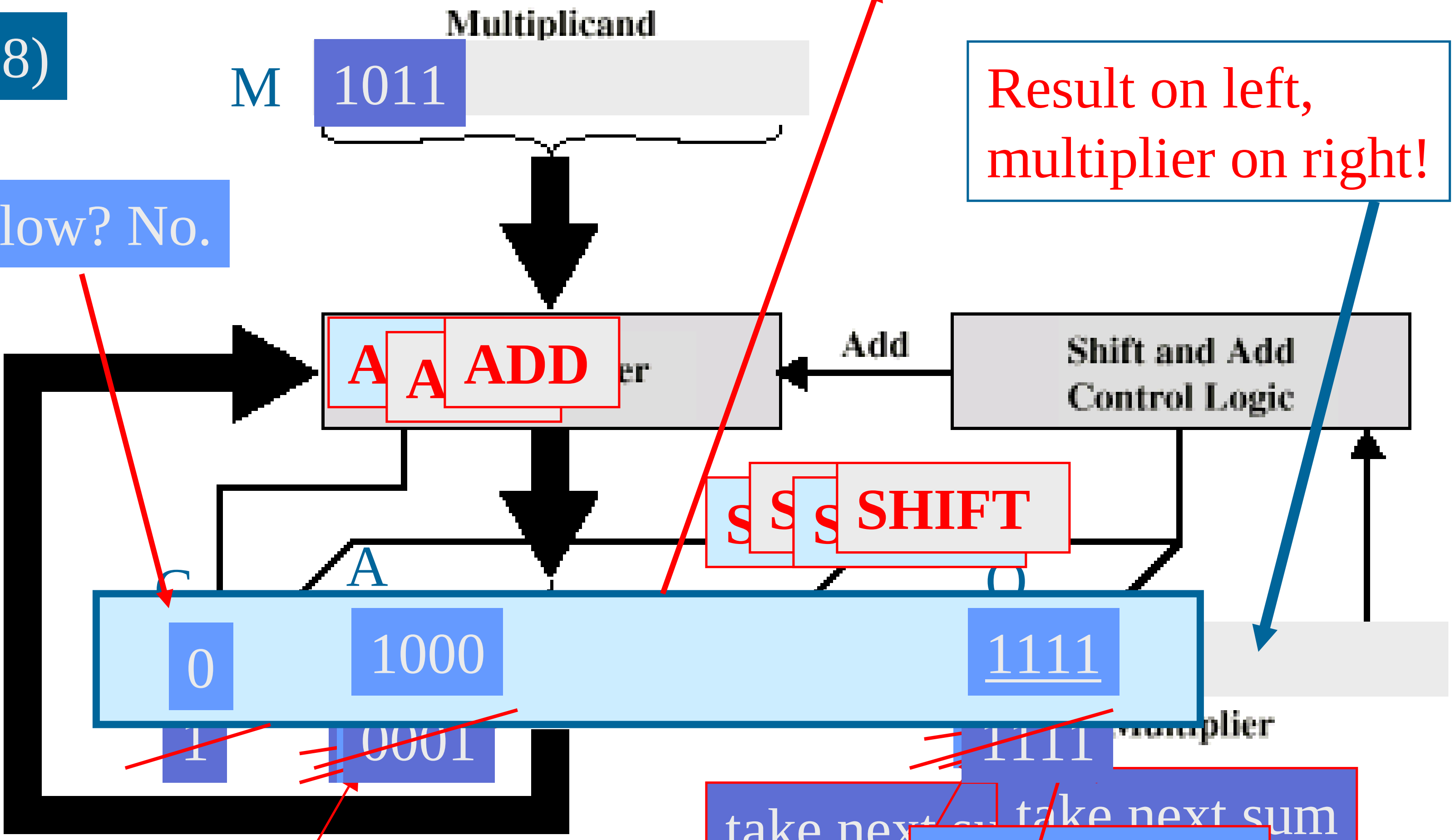
Unsigned Multiplication Example (19)

13 * 11 = ???

= 1000 1111 = 128+8+4+2+1 = 143

(Fig. 8.8)

Overflow? No.



Result on left, multiplier on right!

from C

result bit from A

take next sum

take next sum

just do SHIFT

Multiplication with Negative Values

- Multiplication for unsigned numbers does not work for negative numbers
  - algorithm applies only for unsigned integer representation
  - not the same case as with addition
- Could do it all with unsigned values
  - change operands to positive values
  - do multiplication with positive values
  - negate result if needed
  - OK, but can do better, I.e., faster

The Gist in Booth’s Algorithm (7)

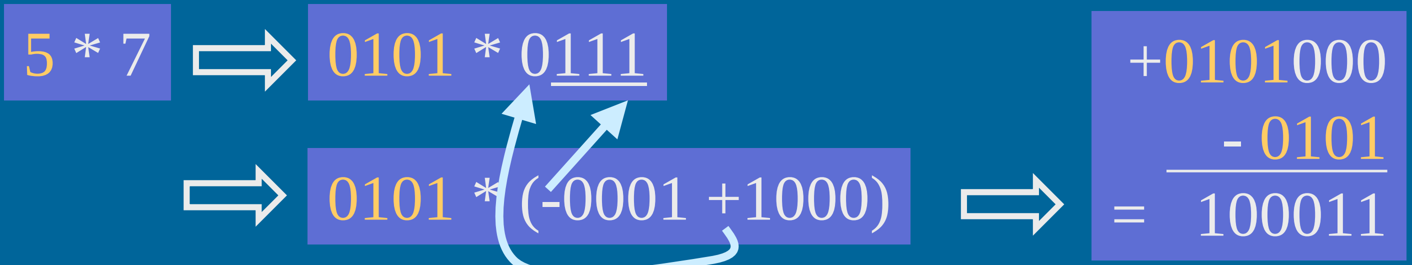
Unsigned multiplication:

addition for every “1” bit
in multiplicand

5 * 7 ⇨ 0101 * 0111 ⇨

  0101
+ 01010
+ 010100
= 100011

- Booth’s algorithm:
  – combine all adjacent 1’s in multiplicand together, replace all additions by one subtraction and one addition (to result)

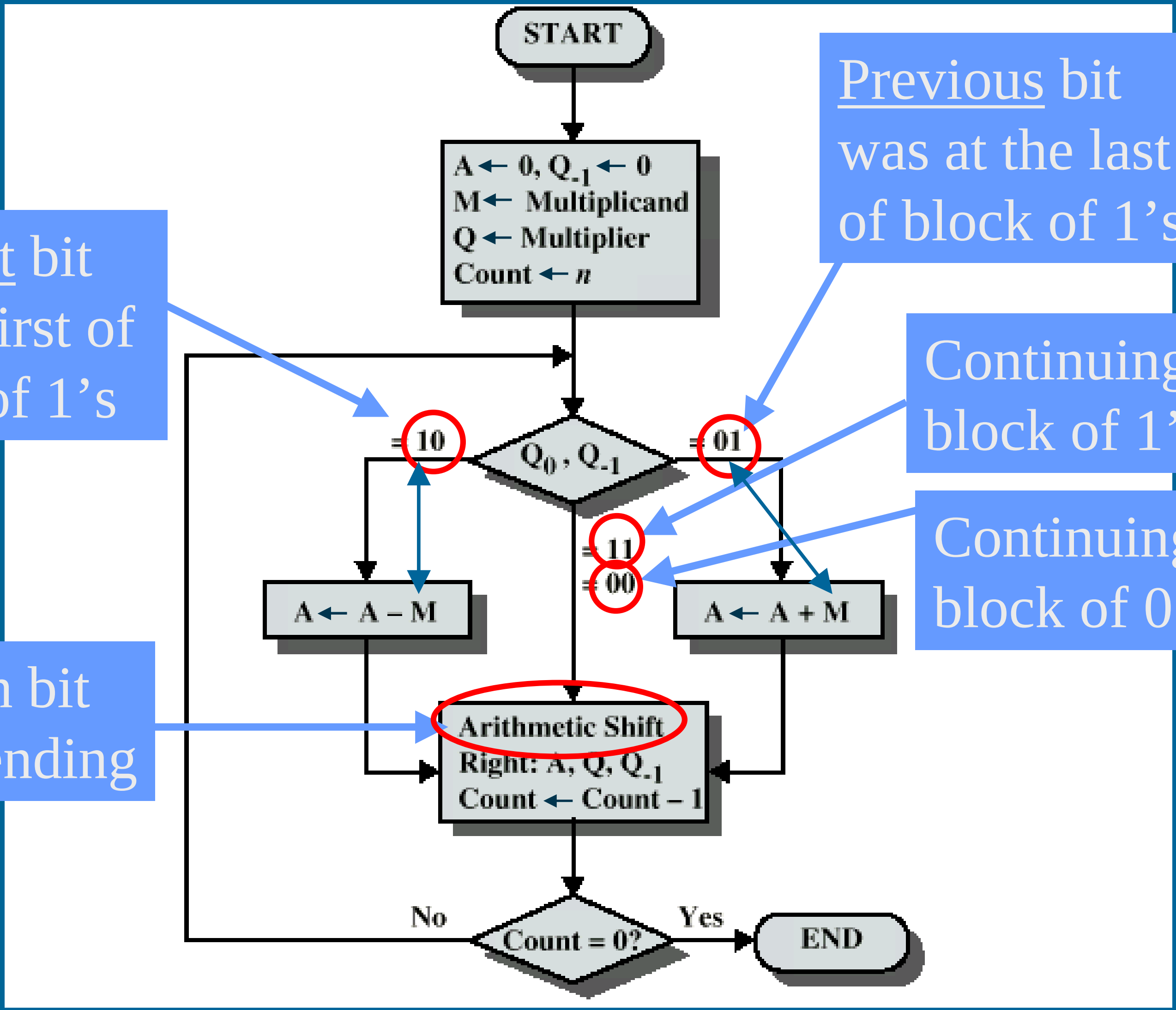


5 * 7 ⇨ 0101 * 0111

⇨ 0101 * (-0001 +1000) ⇨

+0101000
 - 0101
= 100011

Booth’s Algorithm (5)



Current bit is the first of block of 1’s

Previous bit was at the last of block of 1’s

Continuing block of 1’s

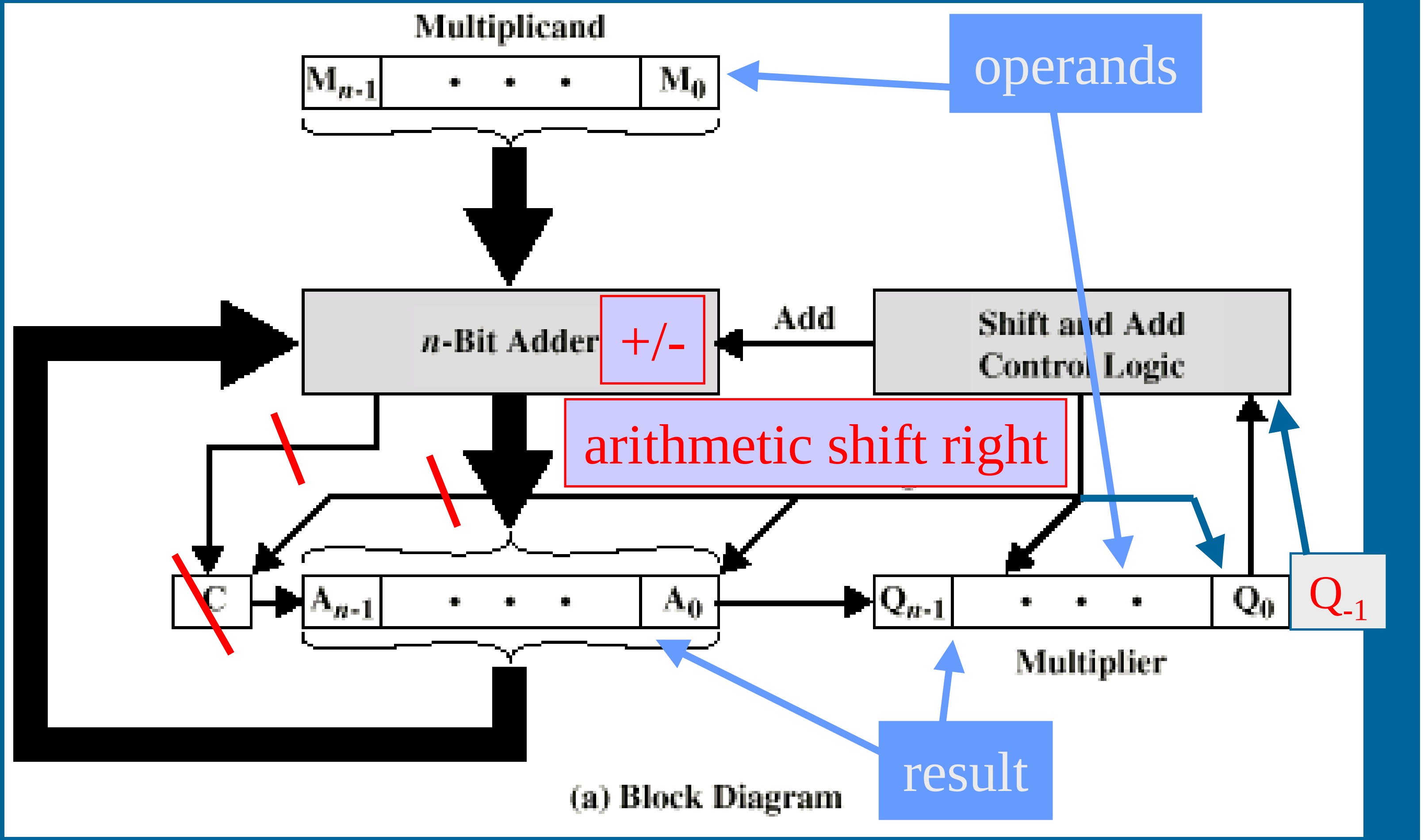
Continuing block of 0’s

Sign bit extending

(Fig. 8.12)

Booth's Algorithm for Twos Complement Multiplication

Fig. 8.12



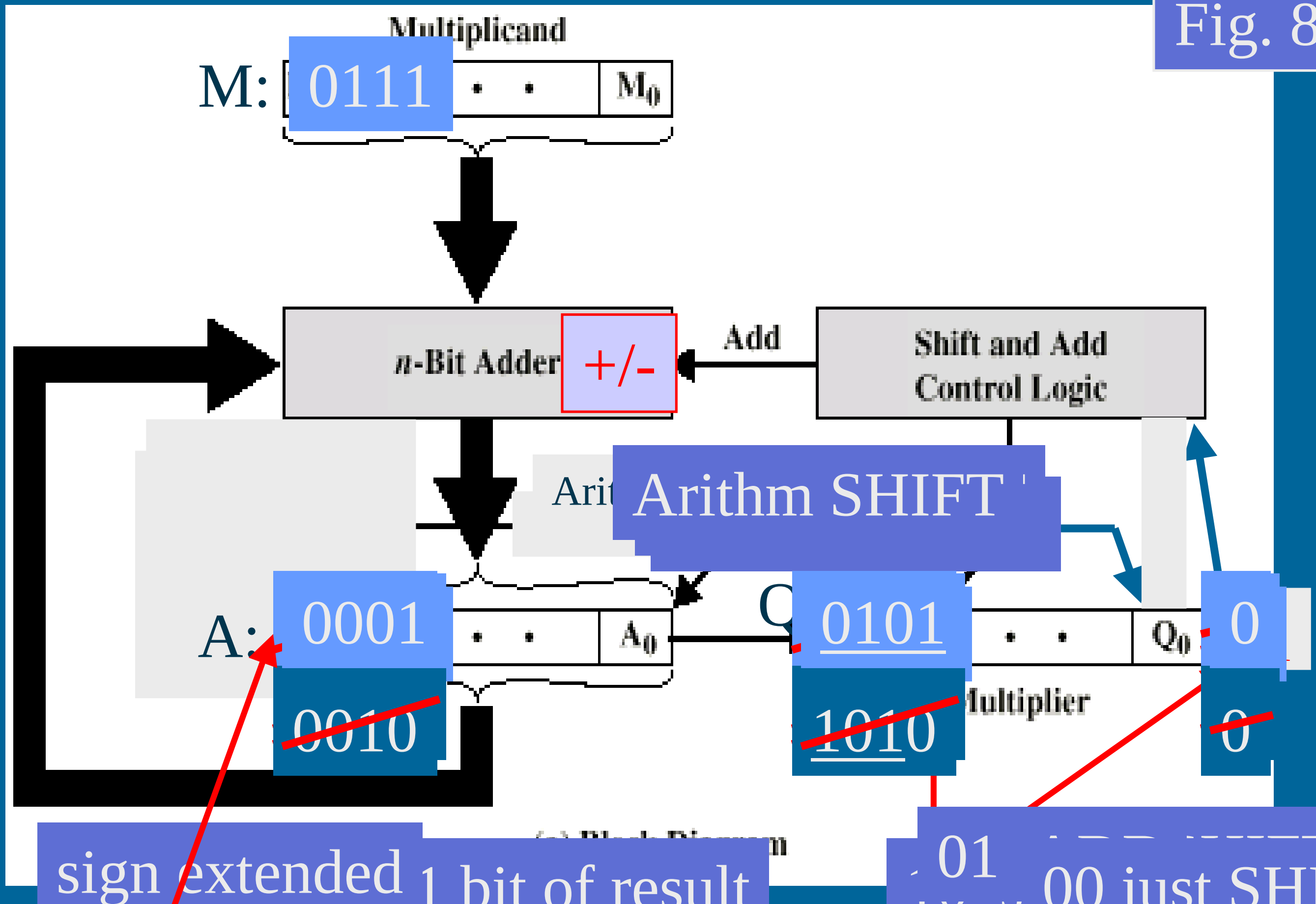
(a) Block Diagram

Booth's Algorithm Example (15)

7 * 3 = ?

= 0001 0101 = 21

Fig. 8.12



Arithm SHIFT

sign extended

1 bit of result

01

00 just SHIFT

Carry bit was lost

Integer Division

- Like in school algorithm

Fig. 8.15

  - easy: new quotient digit 0 or 1
  - M register for dividend (jaettava)
  - Q register for divisor & quotient (jakaja, osamäärä)
  - A register for (partial) remainder (jakojäännös)

Floating Point Representation

$-0.000\ 000\ 000\ 123 = -1.23 * 10^{-10}$

$+0.123 = +1.23 * 10^{-1}$

$+123.0 = +1.23 * 10^{2}$

$+123\ 000\ 000\ 000\ 000 = +1.23 * 10^{14}$

“+”	“14”	“1.23”
sign	exponent	mantissa or significand
	(exponentti)	(mantissa)

IEEE 32-bit Floating Point Standard

IEEE
Standard 754

- 1 bit for sign, 1⇒ "-", 0 ⇒ "+"
- I.e., Stored value S ⇒ Sign value = $(-1)^S$

IEEE 32-bit FP Standard

“+”	“15”	“1.1875” = “1.0011”
sign	exponent	mantissa or significand

- 8 bits for exponent, $2^{8-1}-1 = 127$ biased form

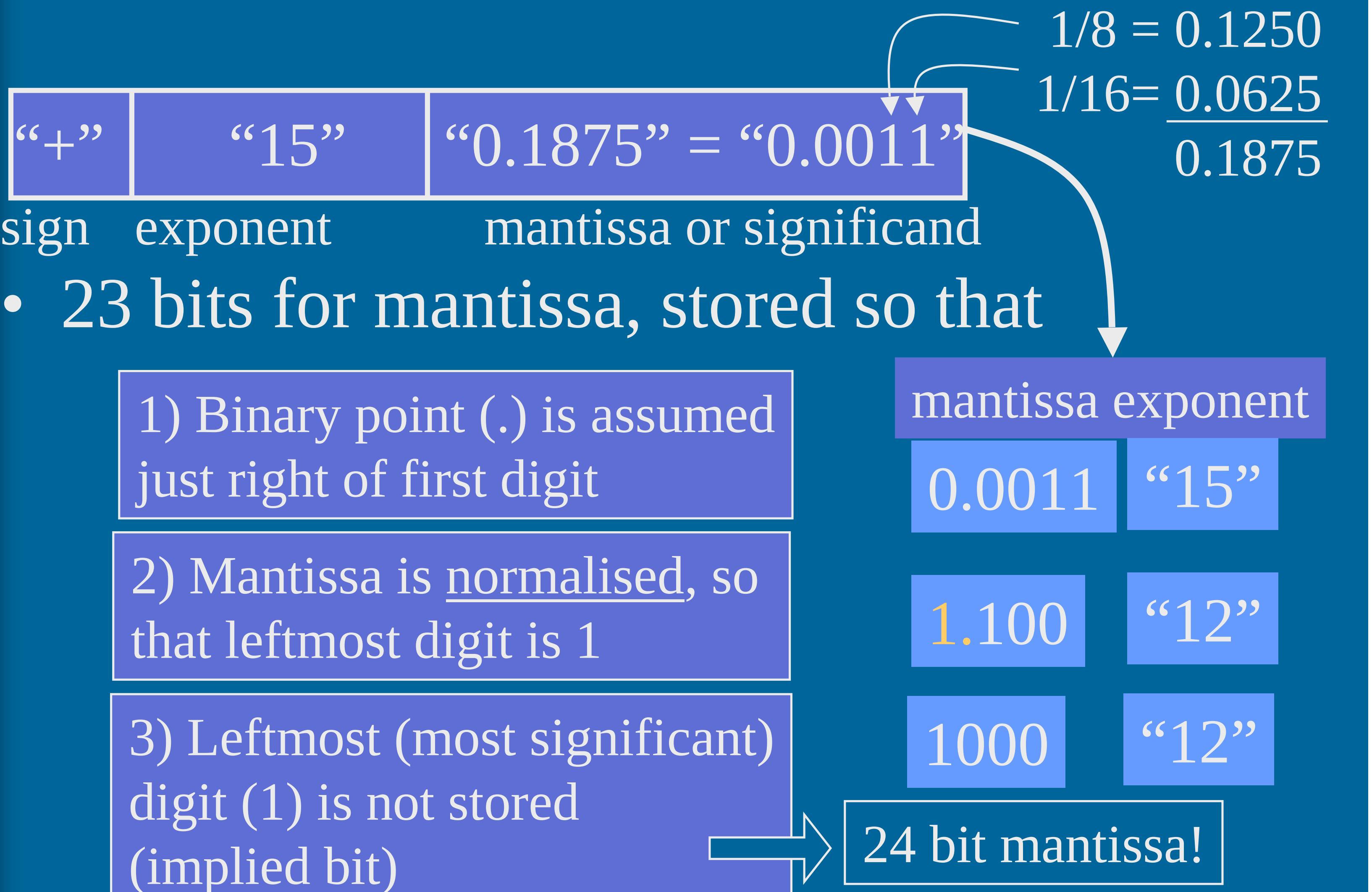
exponent = 5 $\xrightarrow{\text{store}}$ 5+127 = 132 =1000 0100

exponent = -1 $\xrightarrow{\text{store}}$ -1+127 = 126 = 0111 1110

exponent = 0 $\xrightarrow{\text{store}}$ 0+127 = 127 = 0111 1111

  - stored exponents 0 and 255 are special cases
    - stored range: **1 - 254** ⇒ true range: **-126 - 127**

IEEE 32-bit FP Standard (7)



1/8 = 0.1250

1/16= 0.0625

0.1875

"+"	"15"	"0.1875" = "0.0011"
sign	exponent	mantissa or significand

- 23 bits for mantissa, stored so that

1) Binary point (.) is assumed just right of first digit

2) Mantissa is <u>normalised</u>, so that leftmost digit is 1

3) Leftmost (most significant) digit (1) is not stored (implied bit)

mantissa	exponent
0.0011	"15"
1.100	"12"
1000	"12"

24 bit mantissa!

IEEE 32-bit FP Values

$23.0 = +10111.0 * 2^{0} = +1.0111 * 2^{4} =?$

4+127=131

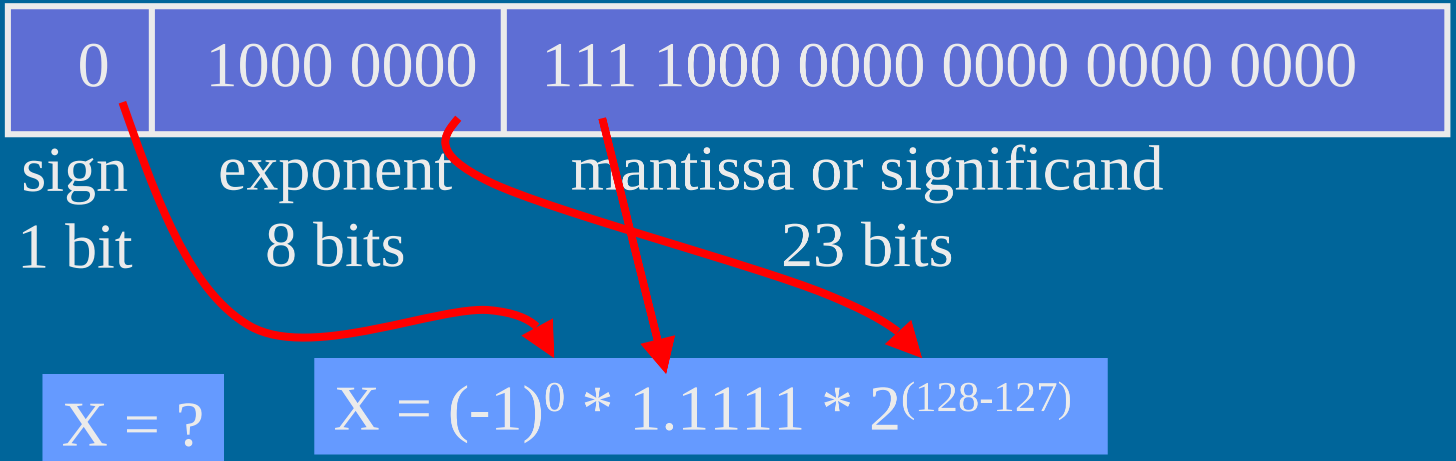
0	1000 0011	011 1000 0000 0000 0000 0000
sign 1 bit	exponent 8 bits	mantissa or significand 23 bits

$1.0 = +1.0000 * 2^{0} = ?$

0+127 = 127

0	0111 1111	000 0000 0000 0000 0000 0000
sign 1 bit	exponent 8 bits	mantissa or significand 23 bits

IEEE 32-bit FP Values



0	1000 0000	111 1000 0000 0000 0000 0000
sign 1 bit	exponent 8 bits	mantissa or significand 23 bits

X = ?

$X = (-1)^0 * 1.1111 * 2^{(128-127)}$

$= 1.1111_2 * 2$

$= (1 + 1/2 + 1/4 + 1/8 + 1/16) * 2$

$= (1 + 0.5 + 0.25 + 0.125 + 0.0625) * 2$

$= 1.9375 * 2$

$= 3.875$

IEEE-754 Floating-Point Conversion

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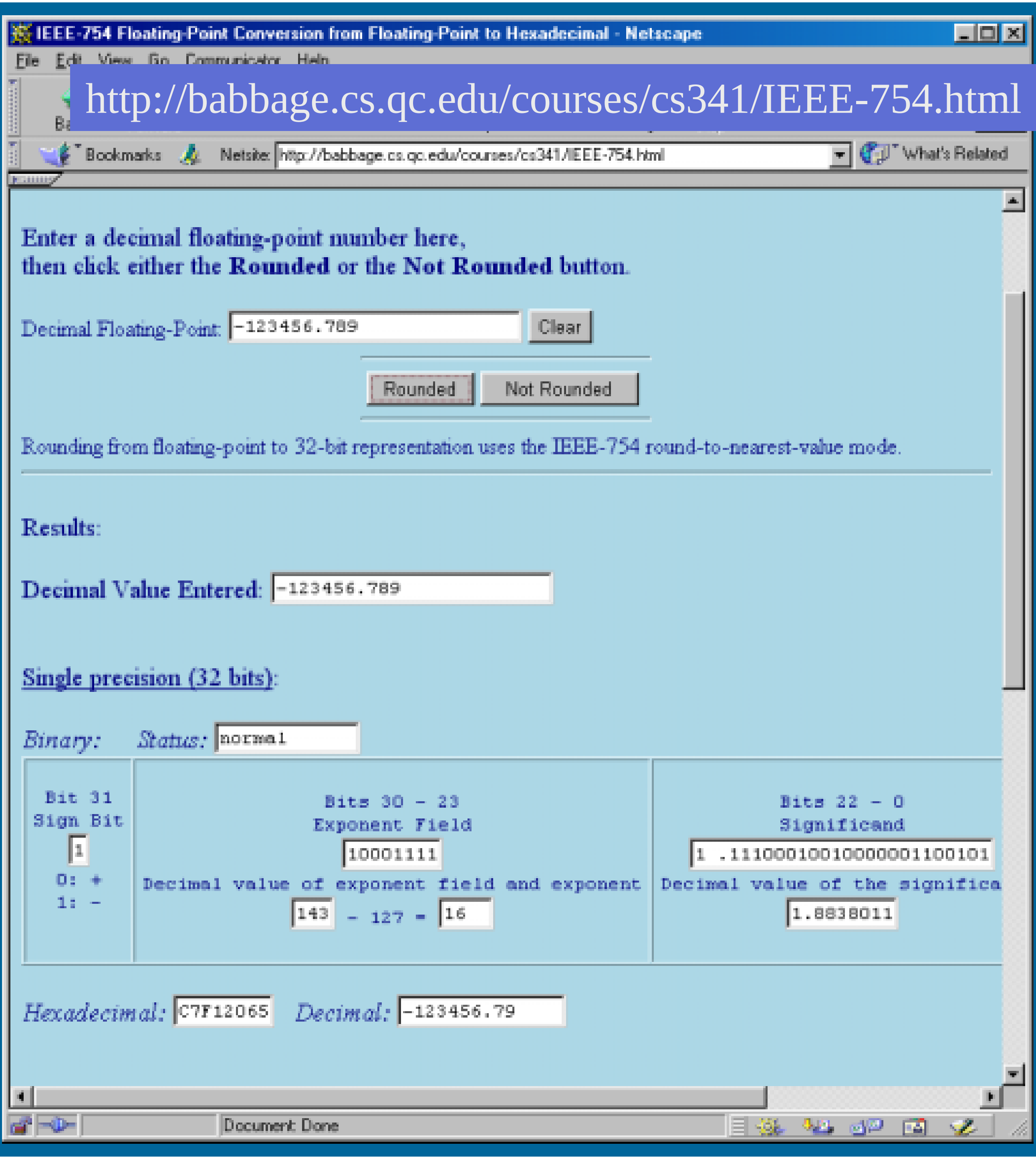
(The City University of New York)

http://babbage.cs.qc.edu/courses/cs341/IEEE-754.html



Enter a decimal floating-point number here, then click either the Rounded or the Not Rounded button.

Decimal Floating-Point: -123456.789

Rounding from floating-point to 32-bit representation uses the IEEE-754 round-to-nearest-value mode.

Results:

Decimal Value Entered: -123456.789

Single precision (32 bits):

Binary: *Status:* normal

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
1	10001111	1 .11100010010000001100101
0: + 1: -	Decimal value of exponent field and exponent 143 - 127 = 16	Decimal value of the significa 1.8838011

Hexadecimal: C7F12065 *Decimal:* -123456.79

IEEE FP Standard

- Single Precision (SP) 32 bits
- Double Precision (DP) 64 bits

(yksin- ja kaksinkertainen tarkkuus)

Table 8.3

- Special values
  - -0, +∞, –∞, NaN
  - denormalized values

Table 8.4

Not a Number

IEEE SP FP Range

- Range
  - 8 bit exponent, effective range: -126 … +127
  - range 2^{-126} ... $2^{127} \approx -10^{-38}$ ... 10^{38}
- Accuracy
  - 23 bit mantissa, 24 bit effective mantissa
  - change least significant digit in mantissa?
  - $2^{24} \approx 1.7 * 10^{-7} \approx$ 6 decimal digits

Floating Point Arithmetic (4)

Table 8.5

- Relatively simple
- Done from internal registers with all bits
  - implied bit included
- Add/subtract
  - more complex than multiplication
  - denormalize first one operand so that both have same exponent
- Multiplication/Division
  - handle mantissa and exponent separately

FP Add or Subtract (4)

- Check for zeroes
  - trivial if one or both operands zero

$1.234 \bullet 10^4$ + $4.444 \bullet 10^6$

- Align mantissas
  - same exponent

$0.01234 \bullet 10^6$ $4.444 \bullet 10^6$

- Add/subtract
  - carry?

    ⇒ shift right and add increase exponent

$4.45634 \bullet 10^6$

- Normalize result
  - shift left, reduce exponent

$4.45634 \bullet 10^6$

FP Special Cases

- Exponent overflow (ylivuoto)
  - above max

Exception Or ±∞ ?

- Exponent underflow (alivuoto)
  - below min

Exception or zero or denormalized?

- Mantissa (significant) underflow
  - in denormalizing may move bits too much right
  - all significant bits lost?

Oooops, lost data!

- Mantissa (significant) overflow

Fix it

  - result of adding mantissas may have carry

FP Multiplication (Division) (7)

Check for zeroes

Result 0, ±∞ ??

Add exponents

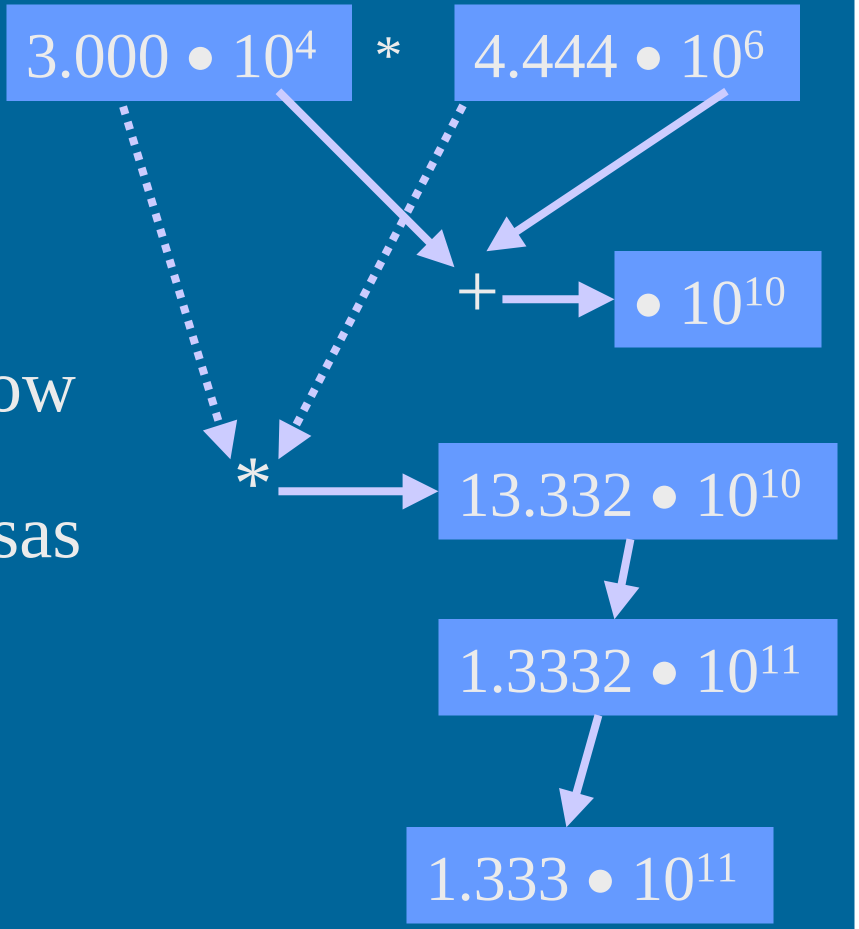
Subtract extra bias

Report overflow/underflow

Multiply (divide) mantissas

Normalise

Round (pyöristä)



$3.000 \bullet 10^4$ * $4.444 \bullet 10^6$

$+ \rightarrow \bullet\ 10^{10}$

$* \rightarrow 13.332 \bullet 10^{10}$

$1.3332 \bullet 10^{11}$

$1.333 \bullet 10^{11}$

Rounding (4)

- Guard bits
  - extra padding with zeroes
  - used with computations only
  - computations with more accuracy than data

$4.444 \bullet 10^6$

$4.444\underline{00} \bullet 10^6$

$$\begin{aligned} 2.0 - 1.9999 &\approx 1.000000 \bullet 2^1 - 0.1111111 \bullet 2^1 \\ &= 1.000000 \bullet 2^1 - 1.111111 \bullet 2^0 \end{aligned}$$

normalised

6 bit mantissa

$$\begin{array}{r} 1.000000 \bullet 2^1 \\ - \; 0.111111 \bullet 2^1 \\ \hline = 0.000001 \bullet 2^1 \\ = 1.000000 \bullet 2^{-5} \end{array}$$

Different accuracy!

$$\begin{array}{r} 1.000000\;00 \bullet 2^1 \\ - \; 0.111111\;10 \bullet 2^1 \\ \hline = 0.000000\;10 \bullet 2^1 \\ = 1.000000\;00 \bullet 2^{-6} \end{array}$$

Align mantis-sas

2 guard bits

Rounding Choices (4)

4 digit accuracy in memory?	3.1234 or -4.5678
● Nearest representable	3.123 or -4.568
● Toward +∞	3.124 or -4.567
● Toward -∞	3.123 or -4.568
● Toward 0	3.123 or -4.567

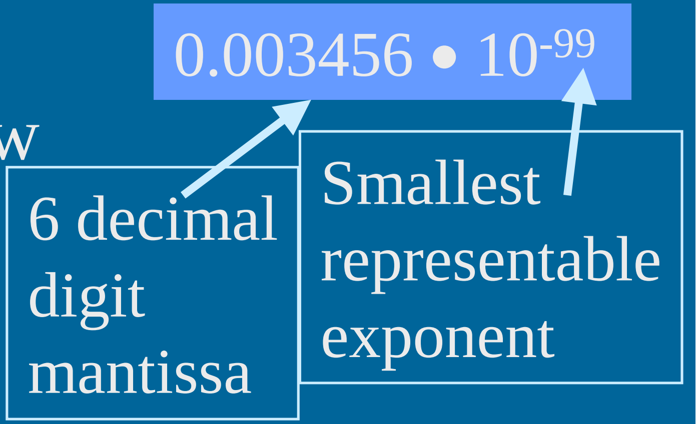
IEEE ∞ and NaN

- ∞
  - outside range of finite numbers
  - rules for arithmetic with ∞: $\infty + \infty = \infty$, etc.
- NaN
  - invalid operation (E.g., 0.0/0.0) can result to NaN or exception
    - user control
    - quiet NaN, or exception?
  - un-initialized data?
  - programming language support?

Table 8.6

IEEE Denormalized Numbers (4)

- Problem: What to do when can not normalize any more?
  - Exponent would underflow

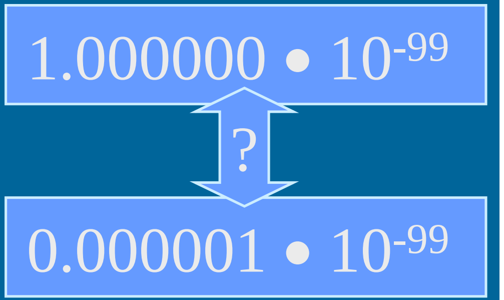


$0.003456 \bullet 10^{-99}$

6 decimal digit mantissa

Smallest representable exponent

- Answer: Denormalized representation
  - smallest representable exponent reserved for this purpose
  - mantissa is not normalized
  - smallest (closest to zero) value is now much smaller than with normalized representation

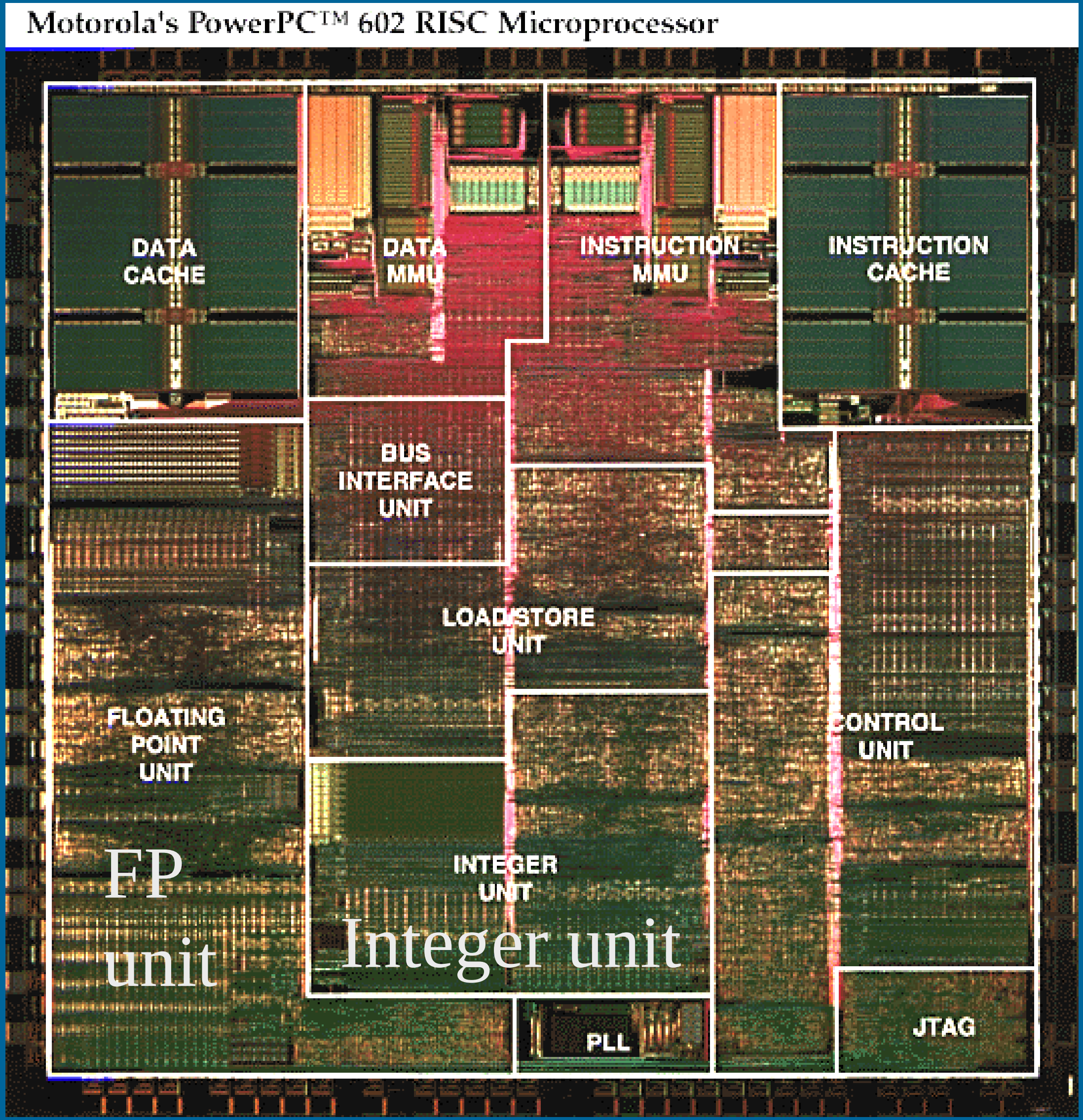


$1.000000 \bullet 10^{-99}$

?

$0.000001 \bullet 10^{-99}$

-- End of Chapter 8: Arithmetic --



http://infopad.eecs.berkeley.edu/CIC/die_photos/