# Finitist Set Theory

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# Abstract

Finitist set theory FST is a maximally simplified set theory for describing relations of finite parts and wholes. Considerable simplifications are achieved by accommodating ur-elements following Kripke-Platek set theory with ur-elements, and by discluding the empty set/the least element following mereology, that exists in other known set theories, while at the same time maintaining inner sets. Having only finite sets is a natural starting point when nothing infinite is needed. Finiteness is natural in respect to real-life problems and computer applications since even a Turing machine has a finite speed and a finite memory tape, and therefore can process only finitely many symbols. The theory of finite sets walks hand in hand with constructivism. According to finitist constructivism, a mathematical object does not exist unless it can be constructed in a finite number of steps.

# 1 Introduction

Starting from the Ancient Greece, the criticism against *actual* infinity or $transfinity$, i.e., infinity in the sense that the series $1, 2, 3, \ldots$ can be considered as a completed totality, has been strong and obvious. Aristotle is the most famous early defender of finitism. Aristotle's finitistic views were unfortunately buried deep along the demolition of his physics and cosmology by the work of Copernicus, Galileo, Kepler, and Newton [8]. The success of Newton's and Leibni's differential and integral calculus demolished Aristotles view that magnitudes such as time and distance cannot be placed into a ratio[1]. Disregarding these flaws, Aristotle's view that the *potential infinity* is totally adequate for the needs of science is a very useful idea. The potential infinity is an extremely simple idea: we can talk about a series $1, 2, 3, \ldots$ as potentially infinite, but not as a completed totality. We can take and use as big numbers as we ever need, but all that we can ever take are finitely big; the series $1, 2, 3, \ldots$ has no upper limit. In other words, we can always take more and more numbers, one after another. Our act of taking or defining the numbers sets the limit for the size of the numbers. This is also the case in FST: there are as many sets as we can say there are, but there will always be only finitely big sets.

Also Ludwig Wittgenstein was a strong proponent of finitism [9, 10, 11]. In [2] Paul Bernays critisized Wittgenstein's [9], but the criticism did not concern his finitist points at all. Crispin Wright's mammoth tome [12] analyzes Wittgenstein's remarks on the foundations of mathematics, but it does not criticize the finitist points either, and does not even discuss transfinite mathematics.

Finitist set theory FST overcomes the conceptual problems and uneconomicality that evolve from transfinity and the empty set. FST is not intended as a foundation of mathematics; this is evident, because finitism is an anti-founationalist approach. FST is intended solely for describing part-whole relations of finite collections, and can be used to model e.g. category structures. FST can be considered also as a minimal axiomatization of 'naive' set theory.

FST uses ideas of *mereology* [7] and *Kripke-Platek set theory with ur-elements* KPU [1]. Following mereology, the axioms of FST disclude the empty set/the least element, which exists generally in set theories such as in KPU, in the de-facto *Zermelo-Fraenkel set theory* ZF, and in *Von Neumann-Gödel-Bernays set theory* NBG [5]. Mereology is unbeatable in its simplicity, but unlike in set theories, mereology has no $\in$ operator, which makes handling inner sets impossible[2]. Following ZF/KPU/NBG, FST accommodates inner sets, which makes FST a richer formalism than mereology. Following KPU and mereology, FST accommodates ur-elements which enables describing atomistic structures more coherently than with ZF/NBG that build all sets on the empty set. Unlike in all the other set theories and in mereology, there can be only a fixed finite number of aggregates/ur-elements and sets in FST.

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[1] In Nichomachean Ethichs, there are examples of having money as a measure of things, which is the same as setting different types of magnitudes into a ratio.

[2] [6] suggests that there is an isomorphism between Boolean algebra without least element and mereology.

# 2 Axioms and Formulas of FST

Axioms of FST define model $\mathcal{L} = \{\mathcal{W}_{\alpha,\beta}, \in\}$. $\mathcal{W}_{\alpha,\beta}$ stands for the allowed sets and ur-elements, where $\alpha \geq 1$ and $\beta \geq 0$. $\alpha$ stands for the number of different types of ur-elements in model $\mathcal{L}$, and $\beta$ stands for the maximum level of rank of the sets of model $\mathcal{L}$. The $memberOf$ operator $\in$ is used to define all the other operators. Symbols $x, y, z$ denote sets, and symbols $r, s, t$ may denote both sets and ur-elements. The symbols for sets may appear on both sides of operators $\subseteq$ and $\in$, but the symbols for ur-elements may appear only on the left side of $\in$[3]. The axioms and formulas of FST are compared to the similar kinds of axioms and formulas of other collection theories in the following.

## 2.1 Axioms of FST

**Axiom of ur-elements**: There exists $\alpha$ ur-elements, where $\alpha \geq 1$. There may exist many instances of the ur-element in more than one context at a given time, i.e., the same ur-element may be a member of more than one set. KPU has a very similar axiom, except that in KPU sets may be formed also without the ur-elements.

**Axiom of extensionality**:

$$\forall x \forall y (\forall r (r \in x \leftrightarrow r \in y) \leftrightarrow x = y).$$

The axiom of extensionality be given in the traditional way. The only difference to ZFC is that also the ur-elements are considered. The axiom of extensionality simply states that a set is determined solely based on its members. With this axiom, reflexivity, transitivity, and symmetricity can be proved. The proofs are given in section 4. A slightly simpler way to give the axiom is Definition 1:

$$\forall r (r \in x \rightarrow r \in y) \equiv x \subseteq y.$$

**Axiom of restriction**:

$$\forall x \exists r (r \in x).$$

Every set must have either a set or an ur-element as a member. By definition, the empty set $\{\}$ has no members, and therefore, there can be no such thing as empty set in FST. Ur-elements are the only $\in$-minimal (epsilon-minimal) elements in FST because they cannot have members.

**Axiom of singleton sets**:

$$\forall r (\mathrm{rank}(r) < \beta \rightarrow \exists x \forall s (s = r \leftrightarrow s \in x)),$$

denoted as $x = \{r\}$. The axiom of pairing that exists e.g. in ZF is unnecessarily strong, and it is adequate to create singleton sets such as $\{r\}$ in order to increase rank. This axiom uses the formula for rank.

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[3] Examples of valid expressions that have a truth-value: $x \subseteq y$, $x \in y$, $r \in y$. When $r \in y$ is true, $r \notin y$ is not true; when $x \subseteq y$ is true, $x \nsubseteq y$ is not true (or $x = y$).

**Axiom of union**:

$$\forall x \forall y \exists z \forall r((r \in x \vee r \in y) \leftrightarrow r \in z),$$

denoted as $z = x \cup y$.

**Axiom of foundation**:

$$\forall x(\exists r(r \in x) \rightarrow \exists s(s \in x \wedge \forall t(t \in x \rightarrow t \notin s))).$$

No set is a member of itself, i.e., that there is no sequence such that $x_{i+1}$ is a member of $x_i$ for all $i$. Within a finite system, if we can proceed from a member to a member of a member indefinitely, we have to confront the same members, and this axiom restricts the infinite chain. This axiom is similar the corresponding axiom of ZF, except that ZF does not consider the difference of sets and ur-elements. Also called the axiom of regularity.

This axiom is is not needed in the sense that the rank-restriction in the axiom of the singleton sets forbids sets that are members of themselves, since those sets surely have a boundless rank, which is again forbidden since a finite $\beta$ must be chosen. However, the rank-restriction only forbids sets that are created *with* the axioms. The axiom of foundation forbids also the paradoxical sets that may have been created *without* the axioms of FST.

## 2.2 Formulas of FST

The formulas of FST are considered as an interface with which FST can be used. The formulas of FST are not used within the axioms except for the formula of rank. Also the formulas of FST differ from the the formulas of other collection theories.

**Disjointness**:

$$\nexists r(r \in x \wedge r \in y),$$

denoted as $x \wr y$. E.g. $\{r\} \wr \{s, t\}$. This way of expressing disjointness is adapted from mereology. This is a remarkable difference towards other set theories. In ZF/NBG/KPU $\{r\} \cap \{s, t\} = \{\}$.

**Proper subset**:

$$x \subseteq y \wedge y \nsubseteq x,$$

denoted as $x \subset y$. E.g. $\{r\} \subset \{r, s\}$. As well as in the formula for disjointness, the use proper subset differs from traditional set theories in respect to the empty set; because it does not exist, it cannot be a proper subset of every other set like in ZF/NBG/KPU. The difference may be explained so that in FST, when $x \subset y$ holds, there must be at least one member in $y$ that is not a member of $x$, all members of $x$ must be members of $y$, and there must be at least one member in $x$, i.e., $x$ cannot be empty. In contrast, in ZF/NBG/KPU it can be considered that when $x \subset y$ holds, then $x$ cannot have any members that are not also members of $y$, and $y$ must have at least one member (that is not a member of $x$). Therefore in ZF/NBG/KPU, $\{\} \subset y$ holds for every $y$ where $y \neq \{\}$. $\{\}$ is a proper subset of every other set in ZF/NBG/KPU because $\{\}$ does not have members that are *not* members of every other set.

**Intersection**:

$$\forall r((r \in x \wedge r \in y) \leftrightarrow r \in z),$$

denoted as $z = x \cap y$. E.g. $\{r, s\} \cap \{s, t\} = \{s\}$. When $r \neq s$, $\{r\} \cap \{s\} = y$ is not true for any $y$.

**Difference**:

$$\forall r(r \in z \leftrightarrow (r \in x \wedge r \notin y)),$$

denoted as $x \backslash y = z$.

**Cardinality**:
card($x$) = 1 is defined as

$$\exists s(s \in x \wedge \forall r(r \in x \leftrightarrow r = s)).$$

card($x$) $\geq n$, where $n \geq 2$ is defined as

$$\exists s_1, s_2, \ldots, s_n((\bigwedge_{k=1}^{n} s_k \in x) \wedge \bigwedge_{a=1}^{n-1} \bigwedge_{b=a+1}^{n} s_a \neq s_b).$$

card($x$) $= n$, where $n \geq 2$ is defined as

$$\mathrm{card}(x) \geq n \wedge \neg(\mathrm{card}(x) \geq n+1).$$

card($x$) gives the number of members of set $x$. E.g. $\mathrm{card}(\{r\}) = 1$, $\mathrm{card}(\{\{r\}\}) = 1$, $\mathrm{card}(\{x, y, r, s\}) = 4$. The lowest possible cardinality of a FST set is 1, when in ZF/NBG/KPU the cardinality of $\{\}$ is 0. Ur-elements do not have cardinality.

**Rank** (used in the axiom of singleton sets):
rank($s$) $= 0 \leftrightarrow s$ is an ur-element.
rank($s$) $\geq n$, where $n \geq 1$, is defined as

$$\exists s_1, s_2, \ldots, s_n(s_1 \in s_2 \in \ldots \in s_n \in s).$$

rank($s$) $= n$, where $n \geq 1$, is defined as

$$\mathrm{rank}(s) \geq n \wedge \neg(\mathrm{rank}(s) \geq n+1).$$

Informally, rank of $s$ stands for the nesting level of $s$, or the greatest length of the recursive chain of inner sets of $s$. When $s$ and $r$ are ur-elements, $\mathrm{rank}(\{s\}) = 1$, $\mathrm{rank}(\{\{s\}\}) = 2$, and $\mathrm{rank}(\{\{s\}, \{\{r\}\}\}) = 3$. Because there is no empty set, the smallest possible rank of any FST set is 1, when in ZF/NBG/KPU the rank of $\{\}$ is 0. Rank could be given also with the recursive formula: $\mathrm{rank}(r) = \sup(\mathrm{rank}(s) + 1 | s \in r)$, where the rank of ur-elements would also be 0.

**Transitivity and transitive closure**:

$$x \subseteq y \wedge \forall r \forall z(r \in z \wedge z \in y \rightarrow r \in y),$$

denoted as $\phi(x, y)$. This means that $x$ is a subset of $y$ and $y$ is transitive towards $x$. In FST every set has transitive closure:

$$\forall x \exists y(\phi(x, y) \wedge \forall z(\phi(x, z) \rightarrow y \subseteq z)),$$

denoted as $y = Tc(x)$. This means that $y$ is the smallest of all possible transitive sets that have $x$ as their subset. Transitive closure of set $x$, $Tc(x)$, contains all members of $x$ and all members of all members of $x$, in all levels of rank, recursively until the $\in$-minimal ur-elements of every member of $x$ are reached. When $r$ and $s$ are ur-elements, e.g. $Tc(\{\{r, s\}\}) = \{r, s, \{r, s\}\}$, $Tc(\{r, \{r, s\}\}) = \{r, s, \{r, s\}\}$, and $Tc(\{\{\{r\}, s\}\}) = \{r, s, \{r\}, \{\{r\}, s\}\}$. Transitive closure in FST is similar than in ZF, except that ZF does not consider the difference of sets and ur-elements.

**Power set**:

$$\forall x(\text{rank}(x) < \beta \rightarrow \exists y \forall z(z \in y \leftrightarrow z \subseteq x)),$$

denoted as $y = P(x)$. Unlike in ZF/NBG/KPU, the power set in FST does not include the empty set. For example, in FST $P(\{r\}) = \{\{r\}\}$, and $P(\{r, s\}) = \{\{r\}, \{s\}, \{r, s\}\}$, but in ZF $P(\{r, s\}) = \{\{\}, \{r\}, \{s\}, \{r, s\}\}$. Therefore, in FST $\text{card}(P(x)) = 2^{\text{card}(x)} - 1$, when in ZF $\text{card}(P(x)) = 2^{\text{card}(x)}$.

# 3 Forming Sets

The ontologist has to first use the axiom of ur-elements and decide the number of ur-elements $\alpha$. Sets of ur-elements may be formed by using the axiom of singleton sets and the axiom of union. Two examples are given of how sets may be formed.

- The goal is to form set $\{r, s\}$. The axiom of ur-elements gives two unidentical ur-elements $r$ and $s$. The axiom of singleton sets gives the singleton sets $\{r\}$ and $\{s\}$. The axiom of union gives the set $\{r\} \cup \{s\} = \{r, s\}$. Set $\{r, s\}$ falls under constraint $\alpha = 2$, $\beta = 1$, i.e., $\mathcal{W}_{2,1}$.
- The goal is to form set $\{r, \{\{s\}\}\}$. The axiom of ur-elements gives the two unidentical ur-elements $r$ and $s$. The axiom of singleton sets gives the singleton sets $\{r\}$ and $\{s\}$. The axiom of singleton sets is applied on set $\{s\}$, which gives $\{\{s\}\}$. When applied on $\{\{s\}\}$, the axiom of singleton sets gives $\{\{\{s\}\}\}$, The axiom of union gives the set $\{r\} \cup \{\{\{s\}\}\} = \{r, \{\{s\}\}\}$. Set $\{r, \{\{s\}\}\}$ falls under constraint $\mathcal{W}_{2,3}$.

All FST sets consists of relations of $\alpha$ ur-elements, and have rank $\beta$ at maximum. The function $sets(\alpha, \beta)$ gives the total number of FST sets with the given $\alpha$ and $\beta$: $sets(\alpha, 0) = 0$. $sets(\alpha, 1) = 2^{\alpha} - 1$. $sets(\alpha, 2) = 2^{sets(\alpha,1)} - 1$. $sets(\alpha, n) = 2^{sets(\alpha,n-1)} - 1$.

# 4 Proofs

In this section, proofs of reflexivity, transitivity, and symmetricity are given.

## 4.1 Proof of Reflexivity

Prove

$$\forall x(x \subseteq x)$$

using the definition 1:

$$\forall r(r \in x \rightarrow r \in y) \equiv x \subseteq y$$

1. Put $x$ also in the place of $y$ in definiton 1.
2. This gives $\forall r(r \in x \rightarrow r \in x) \equiv x \subseteq x$.
3. Because the sentence $\forall r(r \in x \rightarrow r \in x)$ is quantified for every possible set $x$, it can be given as a proposition logical sentence $A \rightarrow A$, which is a tautology that we do not have to prove any more. 4. The step 2. says that this tautology is equivalent with the sentence $x \subseteq x$. Therefore, $\forall x(x \subseteq x)$ holds.

## 4.2 Proof of Symmetricity

Prove

$$\forall x \forall y((x \subseteq y \wedge y \subseteq x) \leftrightarrow x = y)$$

using only:

$$\forall x \forall y(\forall r(r \in x \leftrightarrow r \in y) \leftrightarrow x = y)$$

and the definition 1:

$$\forall r(r \in x \rightarrow r \in y) \equiv x \subseteq y$$

1. We start from the axiom of extensionality: $\forall x \forall y(\forall r(r \in x \leftrightarrow r \in y) \leftrightarrow x = y)$
2. The equivalence can be written as:
$\forall x \forall y((\forall r(r \in x \rightarrow r \in y) \wedge (\forall r(r \in y \rightarrow r \in x))) \leftrightarrow x = y)$

which can be written also as:
$\forall x \forall y(\forall r((r \in x \rightarrow r \in y) \wedge (r \in y \rightarrow r \in x))) \leftrightarrow x = y)$

3. The conjunction holds for all $r$. Therefore
$\forall r((r \in x \rightarrow r \in y) \wedge (r \in y \rightarrow r \in x))$ is equivalent with
$\forall r(r \in x \rightarrow r \in y) \wedge \forall r(r \in y \rightarrow r \in x)$

4. Definition 1 is used, and therefore $\forall r(r \in x \rightarrow r \in y)$ can be abbreviated as $x \subseteq y$:
$\forall x \forall y((x \subseteq y) \wedge (y \subseteq x)) \leftrightarrow x = y)$, which is the definition of symmetricity.

## 4.3 Proof of Transitivity

Prove

$$\forall x \forall y \forall z((x \subseteq y \wedge y \subseteq z) \rightarrow x \subseteq z)$$

using the definition 1:

$$\forall r(r \in x \rightarrow r \in y) \equiv x \subseteq y$$

1. $\forall x \forall y \forall z((x \subseteq y \wedge y \subseteq z) \rightarrow x \subseteq z)$. Using definition 1 gives
2. $\forall x \forall y \forall z[(\forall r(r \in x \rightarrow r \in y) \wedge \forall s(s \in y \rightarrow s \in z)) \rightarrow \forall t(t \in x \rightarrow t \in z)]$
3. As in the proof of symmetricity, the conjuncts in
$(\forall r(r \in x \rightarrow r \in y) \wedge \forall s(s \in y \rightarrow s \in z))$ hold for all $r$ and for all $s$. Therefore 2. can be written as:
$\forall x \forall y \forall z[(\forall r((r \in x \rightarrow r \in y) \wedge (r \in y \rightarrow r \in z)) \rightarrow \forall t(t \in x \rightarrow t \in z)]$
4. And 3. can be reduced to:
$\forall x \forall y \forall z[(\forall r(((r \in x \rightarrow r \in y) \wedge (r \in y \rightarrow r \in z)) \rightarrow (r \in x \rightarrow r \in z))]$
5. Again, because in step 4. the sentence holds for *all* $x, y, z, r$, the sentence in step 4. can be considered to be equivalent with the proposition logical form of transitivity: $(A \rightarrow B \wedge B \rightarrow C) \rightarrow (A \rightarrow C)$, which is proved below. Therefore, transitivity holds in FST.

Here we prove the transitivity as represented in proposition logical form.

Sentence X: $(A \rightarrow B \wedge B \rightarrow C) \rightarrow (A \rightarrow C)$

0. We assume that sentence $X$ is true. Implication $H \rightarrow K$ is not true, only when $K$ is not true while $H$ is true.

1. Similarly, if $(A \rightarrow C)$ is not true while $(A \rightarrow B \wedge B \rightarrow C)$ is true, then X is not true. Therefore, we aim to prove that this cannot be the case.

2. $(A \rightarrow C)$ is not true if and only if $A$ is true while $C$ is not true. So, in order to have X not true, we should have $A$ true while $C$ is not true, simultaneously as having $(A \rightarrow B \wedge B \rightarrow C)$ true. Therefore, we observe the conditions of when $(A \rightarrow B \wedge B \rightarrow C)$ is true.

3. Let us insert the truth-values of $A$ (true) and $C$ (not true) into the sentence $(A \rightarrow B \wedge B \rightarrow C)$: $(true \rightarrow B \wedge B \rightarrow false)$. Now, in order to have $(true \rightarrow B \wedge B \rightarrow false)$ true, $B$ should be true: $(true \rightarrow B)$, and simultaneously $B$ should be false: $(B \rightarrow false)$. Therefore, by the law of contradiction, $(A \rightarrow B \wedge B \rightarrow C)$ cannot be true when $(A \rightarrow C)$ is not true, and therefore the original sentence $X$ is true.

Making the proof even simpler, sentence $X$ can be given in a disjunctive normal form:

0. $(A \rightarrow B \wedge B \rightarrow C) \rightarrow (A \rightarrow B)$
1. $((\neg A \vee B) \wedge (\neg B \vee C)) \rightarrow (\neg A \vee B)$
2. $\neg((\neg A \vee B) \wedge (\neg B \vee C)) \vee (\neg A \vee B)$
3. $(\neg(\neg A \vee B) \vee \neg(\neg B \vee C)) \vee (\neg A \vee B)$
4. $((A \wedge \neg B) \vee (B \wedge \neg C)) \vee (\neg A \vee B)$
5. $(A \wedge \neg B) \vee (B \wedge \neg C) \vee (\neg A \vee B)$

From the disjunctive normal form, a truth table can be built, where $t$ stands for true, and $f$ stands for false.

| $A$ | $B$ | $C$ | $A \wedge \neg B$ | $B \wedge \neg C$ | $\neg A \vee C$ |
|---|---|---|---|---|---|
| $t$ | $t$ | $t$ | $f$ | $f$ | $t$ |
| $t$ | $t$ | $f$ | $f$ | $t$ | $f$ |
| $t$ | $f$ | $t$ | $t$ | $f$ | $t$ |
| $f$ | $t$ | $t$ | $f$ | $f$ | $t$ |
| $f$ | $f$ | $t$ | $f$ | $f$ | $t$ |
| $f$ | $t$ | $f$ | $f$ | $t$ | $t$ |
| $t$ | $f$ | $f$ | $t$ | $f$ | $f$ |
| $f$ | $f$ | $f$ | $f$ | $f$ | $t$ |

In every line, at least one of the disjuncts is true. Therefore, the whole sentence $(A \wedge \neg B) \vee (B \wedge \neg C) \vee (\neg A \vee B)$ is true with every permutation of $A, B, C$.

Now that the proposition logic result has been proved, we can proceed into the axioms of FST, in order to be able to make the similar sort of mapping between the language of FST and proposition logic, as was done in the proof of reflexivity.

There is a difference in proving transitivity and reflexivity from extensionality, compared to proving symmetricity from extensionality. Symmetricity could be proved solely by replacing symbols according to the axioms of FST, without applying the aid of propositional logic.

## 5 Conclusions

The benefits of Finitist set theory as a foundation of ontology, as a collection theory to specify the relations of parts and wholes are considerable towards ZF/NBG/KPU/mereology[4], which is due to finiteness, the ur-elements, and the denial of the empty set. The idea of denial of the empty set was used also in [4].

FST can be taken simply as a finite version of KPU without the empty set, or as a finite version of discrete (atomistic) mereology with inner sets, with the exception that larger collections may not be formed of the ur-elements themselves, without the sets. FST accommodates ur-elements from KPU and from disrete mereology, inner sets from set theories in general, and infiniteness is totally banned to start with. Having inner sets makes FST a richer collection theory than mereology, and yet the axiomatization of FST is very much simpler than that of ZF/NBG/KPU because of the denial of infinity and the denial of the empty set. The axiom of foundation/regularity could be discluded because irregular sets cannot be formed with the axioms, and irregular sets would have an infinite rank. Power set does not need to be an axiom either because it is not necessary in order to form the needed sets. There is naturally no need for the axiom of infinity, not to mention the axiom of choice. Also, the holding of the law of the excluded middle is naturally not even a question within the finite domain of FST, because the debate on it concerns only situations where it is not surveyable or provable whether $A$ holds, or $\neg A$ holds. The law of the excluded middle holds in FST, because proving any theorem that can be stated under the domain of FST requires only a finite number of steps.

Infinitely large sets do not bother FST, and yet FST offers sufficient tools to describe finite structures. As a finite system, FST suits especially for modeling information systems such as file structures better than other set theories to start with, and FST can

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[4] See http://www.cs.helsinki.fi/u/astyrman/set.pdf for an explanation of some of the ontological benefits of FST.

be adjusted depending on the usage. Understanding FST is very much easier and far more intuitive than understanding other known set theories. FST can be used in modeling ontology in simpler means that with any other collection theory that accommodates inner sets..

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