Fte263 / 582418 Proof Theory and Proof Search, spring 2004. Exercises 5

1. Download and install a proof editor (either PESCA or ProEd). You will find links to PESCA and ProEd at the course home page. PESCA requires that you have Haskell installed in your system and ProEd requires that you have Java. Java is installed in most university computers but Haskell can probably be found only in the Linux systems of the Department of Computer Science. Both programming languages are freely available so you can download and install them to your computer if needed. The links can be found at the course home page.

2. Use a proof editor to give sequent calculus proof in **G3cp** of $\Rightarrow D$, where D is

   (a) $(A \supset B \vee C) \supset (A \supset B) \vee (A \supset C)$

   (b) $((A \,\&\, {\sim}B) \,\&\, ((((A \vee (B \supset B)) \supset {\sim}(A \,\&\, B)) \vee {\sim}B) \supset {\sim}A)) \supset C$

   (c) ${\sim}((((A \supset A) \supset A) \,\&\, (({\sim}A \supset A) \supset {\sim}A)) \vee (((B \supset B) \supset B) \,\&\, {\sim}B) \vee {\sim}{\sim}({\sim}(A \supset B) \,\&\, {\sim}(B \supset {\sim}{\sim}A)))$

3. Complete the proof of the admissibility of cut (Theorem 3.2.3, page 54) in the following cases:

   (a) The right premiss is an axiom or conclusion of $L\bot$.

   (b) The cut formula is not principal in the left premiss and the left premiss is derived by $L\vee$

   (c) The cut formula of the form $A \vee B$ or $A \,\&\, B$ is principal in both premisses.

4. Consider the quantifier rules given in the next page and explain what is wrong in the following derivations

$$\dfrac{\dfrac{[\exists xA]_2 \qquad \dfrac{[A(y/x)]_1}{\forall xA}}{\forall xA}\,{\scriptstyle \exists E_1}}{\exists xA \supset \forall xA}\,{\scriptstyle \supset I_2}
\qquad
\dfrac{\dfrac{\dfrac{[\exists xA]_2 \qquad \dfrac{\dfrac{[\forall x(A \supset B)]_3}{A(y/x) \supset B}\,{\scriptstyle \forall E} \qquad [A(y/x)]_1}{B}\,{\scriptstyle \supset E}}{B}\,{\scriptstyle \exists E_1}}{\exists xA \supset B}\,{\scriptstyle \supset I_2}}{\forall x(A \supset B) \supset (\exists xA \supset B)}\,{\scriptstyle \supset I_3}$$

   Does the additional hypothesis that B does not contain any free variable suffice to make the second one correct?

5. Consider a language without constants, nor functions. Prove that if $\exists xA$ is derivable, then $\forall xA$ is derivable.

6. Why are the following derivations incorrect?

$$\dfrac{\forall x \exists y(y = x+1)}{\exists y(y = y+1)}\,{\scriptstyle \forall E}
\qquad
\dfrac{\sim \exists x(x \neq x)}{\exists y(\sim \exists x(x \neq y))}\,{\scriptstyle \exists I}$$

The **introduction rules** for the quantifiers are

$$\frac{A(y/x)}{\forall x A}\,\forall I \qquad \frac{A(t/x)}{\exists x A}\,\exists I$$

The rule of universal introduction has the variable restriction that y must not occur free in any assumption that $A(y/x)$ depends on, nor in $\forall x A$. In rule $\exists I$, t is a term free for x in A.

The **elimination rules** are

$$\frac{\forall x A}{A(t/x)}\,\forall E \qquad \frac{\exists x A \qquad \begin{array}{c}[A(y/x)]\\ \vdots \\ C\end{array}}{C}\,\exists E$$

The latter has the restriction that y must not occur free in $\exists x A$, C, nor in any assumption C depends on except $A(y/x)$. In the former, t is a term free for x in A.