Fte263 / 582418 Proof Theory, spring 2004. Solutions 1

1. Prove in natural deduction for minimal logic (that is, with introduction and elimination rules for the connectives except $\bot E$):

   (a) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \supset C]^2 \qquad \dfrac{[A \supset B]^3 \quad [A]^1}{B}\,{\supset}E}{C}\,{\supset}E}{A \supset C}\,{\supset}I,1}{(B \supset C) \supset (A \supset C)}\,{\supset}I,2}{(A \supset B) \supset ((B \supset C) \supset (A \supset C))}\,{\supset}I,3$$

   (b) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \supset (B \supset C)]^3 \quad [A]^1}{B \supset C}\,{\supset}E \qquad \dfrac{[A \supset B]^2 \quad [A]^1}{B}\,{\supset}E}{C}\,{\supset}E}{A \supset C}\,{\supset}I,1}{(A \supset B) \supset (A \supset C)}\,{\supset}I,2}{(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))}\,{\supset}I,3$$

   (c) $(A \& B \supset C) \supset\subset (A \supset (B \supset C))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \& B \supset C]^3 \qquad \dfrac{[A]^2 \quad [B]^1}{A \& B}\,\&I}{C}\,{\supset}E}{B \supset C}\,{\supset}I,1}{A \supset (B \supset C)}\,{\supset}I,2}{(A \& B \supset C) \supset (A \supset (B \supset C))}\,{\supset}I,3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[A \supset (B \supset C)]^5 \qquad \dfrac{[A \& B]^4}{A}\,\&E}{B \supset C}\,{\supset}E \qquad \dfrac{[A \& B]^4}{B}\,\&E}{C}\,{\supset}E}{A \& B \supset C}\,{\supset}I,4}{(A \supset (B \supset C)) \supset (A \& B \supset C)}\,{\supset}I,5}{(A \& B \supset C) \supset (A \supset (B \supset C)) \,\&\, (A \supset (B \supset C)) \supset (A \& B \supset C)}\,\&I$$

2. Prove in natural deduction for minimal logic:

   (a) $(A \supset B) \supset (\sim B \supset \sim A)$ (contraposition)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\sim B]^2 \qquad \dfrac{[(A \supset B]^3 \quad [A]^1}{B}\,{\supset}E}{\bot}\,{\supset}E}{\sim A}\,{\supset}I,1}{\sim B \supset \sim A}\,{\supset}I,2}{(A \supset B) \supset (\sim B \supset \sim A)}\,{\supset}I,3$$

   (b) $\sim(A \vee B) \supset\subset (\sim A \,\&\, \sim B)$ (de Morgan's laws)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\sim(A \vee B)]^3 \qquad \dfrac{[A]^1}{A \vee B}\,{\vee}I}{\bot}\,{\supset}E}{\sim A}\,{\supset}I,1 \qquad \dfrac{\dfrac{[\sim(A \vee B)]^3 \qquad \dfrac{[B]^2}{A \vee B}\,{\vee}I}{\bot}\,{\supset}E}{\sim B}\,{\supset}I,2}{\sim A \,\&\, \sim B}\,\&I}{\sim(A \vee B) \supset (\sim A \,\&\, \sim B)}\,{\supset}I,3 \qquad \dfrac{\dfrac{\dfrac{[A \vee B]^5 \qquad \dfrac{[A]^4 \qquad \dfrac{[\sim A \,\&\, \sim B]^6}{\sim A}\,\&E}{\bot}\,{\supset}E \qquad \dfrac{[B]^4 \qquad \dfrac{[\sim A \,\&\, \sim B]^3}{\sim B}\,\&E}{\bot}\,{\supset}E}{\bot}\,{\vee}E,4}{\sim(A \vee B)}\,{\supset}I,5}{(\sim A \,\&\, \sim B) \supset \sim(A \vee B)}\,{\supset}I,6}{(\sim(A \vee B) \supset (\sim A \,\&\, \sim B)) \,\&\, ((\sim A \,\&\, \sim B) \supset \sim(A \vee B))}\,\&I$$

$\sim A \vee \sim B \supset \sim(A \& B)$

$$\dfrac{\dfrac{[\sim A \vee \sim B]^4 \qquad \dfrac{\dfrac{[\sim A]^3 \quad \dfrac{[A \& B]^1}{A}{\scriptstyle\,\&E}}{\bot}{\scriptstyle\,\supset E}}{\sim(A \& B)}{\scriptstyle\,\supset I,1} \qquad \dfrac{\dfrac{[\sim B]^3 \quad \dfrac{[A \& B]^2}{B}{\scriptstyle\,\&E}}{\bot}{\scriptstyle\,\supset E}}{\sim(A \& B)}{\scriptstyle\,\supset I,2}}{\sim(A \& B)}{\scriptstyle\,\vee E,3}}{(\sim A \vee \sim B) \supset \sim(A \& B)}{\scriptstyle\,\supset I,4}$$

(c) $A \vee B \supset \sim(\sim A \& \sim B)$

$$\dfrac{\dfrac{\dfrac{[A \vee B]^3 \qquad \dfrac{[A]^1 \quad \dfrac{[\sim A \& \sim B]^2}{\sim A}{\scriptstyle\,\&E}}{\bot}{\scriptstyle\,\supset E} \qquad \dfrac{[B]^1 \quad \dfrac{[\sim A \& \sim B]^2}{\sim B}{\scriptstyle\,\&E}}{\bot}{\scriptstyle\,\supset E}}{\bot}{\scriptstyle\,\vee E,1}}{\sim(\sim A \& \sim B)}{\scriptstyle\,\supset I,2}}{A \vee B \supset \sim(\sim A \& \sim B)}{\scriptstyle\,\supset I,3}$$

(d) $\sim(A \& B) \supset\subset (A \supset \sim B)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\sim(A \& B)]^3 \quad \dfrac{[A]^2 \quad [B]^1}{A \& B}{\scriptstyle\,\&I}}{\bot}{\scriptstyle\,\supset E}}{\sim B}{\scriptstyle\,\supset I,1}}{A \supset \sim B}{\scriptstyle\,\supset I,2}}{\sim(A \& B) \supset (A \supset \sim B)}{\scriptstyle\,\supset I,3} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[A \supset \sim B]^5 \quad \dfrac{[A \& B]^4}{A}{\scriptstyle\,\&E}}{\sim B}{\scriptstyle\,\supset E} \quad \dfrac{[A \& B]^4}{B}{\scriptstyle\,\&E}}{\bot}{\scriptstyle\,\supset E}}{\sim(A \& B)}{\scriptstyle\,\supset I,4}}{(A \supset \sim B) \supset \sim(A \& B)}{\scriptstyle\,\supset I,5}}{\sim(A \& B) \supset\subset (A \supset \sim B)}{\scriptstyle\,\&I}$$

3. Prove in natural deduction for intuitionistic logic (that is, without excluded middle or its special case *reductio ad absurdum*):

(a) $(A \& \sim A) \supset B$

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \& \sim A]^1}{\sim A}{\scriptstyle\,\&E} \quad \dfrac{[A \& \sim A]^1}{A}{\scriptstyle\,\&E}}{\bot}{\scriptstyle\,\supset E}}{B}{\scriptstyle\,\bot E}}{(A \& \sim A) \supset B}{\scriptstyle\,\supset I,1}$$

(b) $\sim(A \supset B) \supset \sim\sim A \& \sim B$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\sim(A \supset B)]^4 \quad \dfrac{\dfrac{\dfrac{[\sim A]^2 \quad [A]^1}{\bot}{\scriptstyle\,\supset E}}{B}{\scriptstyle\,\bot E}}{A \supset B}{\scriptstyle\,\supset I,1}}{\bot}{\scriptstyle\,\supset E}}{\sim\sim A}{\scriptstyle\,\supset I,2} \qquad \dfrac{\dfrac{[\sim(A \supset B)]^4 \quad \dfrac{[B]^3}{A \supset B}{\scriptstyle\,\supset I}}{\bot}{\scriptstyle\,\supset E}}{\sim B}{\scriptstyle\,\supset I,3}}{\sim\sim A \& \sim B}{\scriptstyle\,\&I}}{\sim(A \supset B) \supset \sim\sim A \& \sim B}{\scriptstyle\,\supset I,4}$$

(c) $\sim A \vee B \supset (A \supset B)$

$$\dfrac{[\sim A \vee B]^3 \qquad \dfrac{\dfrac{\dfrac{[\sim A]^2 \quad [A]^1}{\bot}{\scriptstyle \supset E}}{B}{\scriptstyle \bot E}}{A \supset B}{\scriptstyle \supset I,1} \qquad \dfrac{[B]^2}{A \supset B}{\scriptstyle \supset I}}{\dfrac{A \supset B}{\sim A \vee B \supset (A \supset B)}{\scriptstyle \supset I,3}}{\scriptstyle \vee E,2}$$

4. (Gödel brain test) Prove in natural deduction for minimal logic, in less than 2 minutes for each direction: $\sim\sim(A \,\&\, B) \supset\subset \sim\sim A \,\&\, \sim\sim B$.

$$\dfrac{\dfrac{\dfrac{[\sim\sim(A \,\&\, B)]^5 \qquad \dfrac{\dfrac{\dfrac{[A \,\&\, B]^1}{A}{\scriptstyle \&E} \quad [\sim A]^3}{\bot}{\scriptstyle \supset E}}{\sim(A \,\&\, B)}{\scriptstyle \supset I,1}}{\bot}{\scriptstyle \supset E}}{\sim\sim A}{\scriptstyle \supset I,3} \qquad \dfrac{\dfrac{[\sim\sim(A \,\&\, B)]^5 \qquad \dfrac{\dfrac{\dfrac{[A \,\&\, B]^2}{B}{\scriptstyle \&E} \quad [\sim B]^4}{\bot}{\scriptstyle \supset E}}{\sim(A \,\&\, B)}{\scriptstyle \supset I,2}}{\bot}{\scriptstyle \supset E}}{\sim\sim B}{\scriptstyle \supset I,4}}{\dfrac{\sim\sim A \,\&\, \sim\sim B}{\sim\sim(A \,\&\, B) \supset (\sim\sim A \,\&\, \sim\sim B)}{\scriptstyle \supset I,5}}{\scriptstyle \&I}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\sim(A \,\&\, B)]^8 \qquad \dfrac{[A]^6 \quad [B]^7}{A \,\&\, B}{\scriptstyle \&I}}{\bot}{\scriptstyle \supset E}}{\sim A}{\scriptstyle \supset I,6} \qquad \dfrac{[\sim\sim A \,\&\, \sim\sim B]^9}{\sim\sim A}{\scriptstyle \&E}}{\bot}{\scriptstyle \supset E}}{\sim B}{\scriptstyle \supset I,7} \qquad \dfrac{[\sim\sim A \,\&\, \sim\sim B]^9}{\sim\sim B}{\scriptstyle \&E}}{\bot}{\scriptstyle \supset E}}{\dfrac{\sim\sim(A \,\&\, B)}{\sim\sim A \,\&\, \sim\sim B \supset \sim\sim(A \,\&\, B)}{\scriptstyle \supset I,9}}{\scriptstyle \supset I,8}$$