Fte263 / 582418 Proof Theory, spring 2004. Solutions 2

1. Prove in natural deduction for minimal logic using the general elimination rules:

   (a) $A \& B \supset\subset B \& A$ (commutativity)

$$\dfrac{\dfrac{\dfrac{[A\&B]^2 \quad \dfrac{[B]^1 \quad [A]^1}{B\&A}{\scriptstyle \&I}}{B\&A}{\scriptstyle \&E,1}}{(A\&B)\supset(B\&A)}{\scriptstyle \supset I,2} \qquad \dfrac{\dfrac{[B\&A]^4 \quad \dfrac{[A]^3 \quad [B]^3}{A\&B}{\scriptstyle \&I}}{A\&B}{\scriptstyle \&E,3}}{(B\&A)\supset(A\&B)}{\scriptstyle \supset I,4}}{((A\&B)\supset(B\&A))\,\&\,((B\&A)\supset(A\&B))}{\scriptstyle \&I}$$

   $A \vee B \supset\subset B \vee A$

$$\dfrac{\dfrac{\dfrac{[A\vee B]^2 \quad \dfrac{[A]^1}{B\vee A}{\scriptstyle \vee I_2} \quad \dfrac{[B]^1}{B\vee A}{\scriptstyle \vee I_1}}{B\vee A}{\scriptstyle \vee E,1}}{(A\vee B)\supset(B\vee A)}{\scriptstyle \supset I,2} \qquad \dfrac{\dfrac{[B\vee A]^4 \quad \dfrac{[A]^3}{A\vee B}{\scriptstyle \vee I_1} \quad \dfrac{[B]^3}{A\vee B}{\scriptstyle \vee I_2}}{A\vee B}{\scriptstyle \vee E,3}}{(B\vee A)\supset(A\vee B)}{\scriptstyle \supset I,4}}{((A\vee B)\supset(B\vee A))\,\&\,((B\vee A)\supset(A\vee B))}{\scriptstyle \&I}$$

   (b) $(A \& B \supset C) \supset\subset (A \supset (B \supset C))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A\&B\supset C]^4 \quad \dfrac{[A]^2 \quad [B]^2}{A\&B}{\scriptstyle \&I} \quad [C]^1}{C}{\scriptstyle \supset E,1}}{B\supset C}{\scriptstyle \supset I,2}}{A\supset(B\supset C)}{\scriptstyle \supset I,3}}{(A\&B\supset C)\supset(A\supset(B\supset C))}{\scriptstyle \supset I,4} \qquad \dfrac{\dfrac{\dfrac{[A\supset(B\supset C)]^{10} \quad \dfrac{[A\&B]^9 \quad [A]^5}{A}{\scriptstyle \&E,5} \quad \dfrac{[B\supset C]^8 \quad \dfrac{[A\&B]^9 \quad [B]^6}{B}{\scriptstyle \&E,6} \quad [C]^6}{C}{\scriptstyle \supset E,7}}{C}{\scriptstyle \supset E,8}}{A\&B\supset C}{\scriptstyle \supset I,9}}{A\supset(B\supset C)\supset(A\&B\supset C)}{\scriptstyle \supset I,10}}{(A\&B\supset C)\supset(A\supset(B\supset C))\,\&\,(A\supset(B\supset C))\supset(A\&B\supset C)}{\scriptstyle \&I}$$

2. Prove in natural deduction for minimal logic using the general elimination rules:

   (a) $\sim(A \& \sim A)$

$$\dfrac{\dfrac{[A \& \sim A]^3 \quad \dfrac{[\sim A]^2 \quad [A]^2 \quad [\bot]^1}{\bot}{\scriptstyle \supset E,1}}{\bot}{\scriptstyle \&E,2}}{\sim(A \& \sim A)}{\scriptstyle \supset I,3}$$

   (b) $A \supset \sim\sim A$

$$\dfrac{\dfrac{\dfrac{[\sim A]^2 \quad [A]^3 \quad [\bot]^1}{\bot}{\scriptstyle \supset E,1}}{\sim\sim A}{\scriptstyle \supset I,2}}{A \supset \sim\sim A}{\scriptstyle \supset I,3}$$

   (c) $\sim\sim\sim A \supset\subset \sim A$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\sim\sim\sim A]^5 \quad \dfrac{\dfrac{[\sim A]^2 \quad [A]^4 \quad [\bot]^1}{\bot}{\scriptstyle \supset E,1}}{\sim\sim A}{\scriptstyle \supset I,2} \quad [\bot]^3}{\bot}{\scriptstyle \supset E,3}}{\sim A}{\scriptstyle \supset I,4}}{\sim\sim\sim A \supset \sim A}{\scriptstyle \supset I,5} \qquad \dfrac{\dfrac{\dfrac{[\sim\sim A]^7 \quad [\sim A]^8 \quad [\bot]^6}{\bot}{\scriptstyle \supset E,6}}{\sim\sim\sim A}{\scriptstyle \supset I,7}}{\sim A \supset \sim\sim\sim A}{\scriptstyle \supset I,8}}{(\sim\sim\sim A \supset \sim A)\,\&\,(\sim A \supset \sim\sim\sim A)}{\scriptstyle \&E}$$

(d) $\sim\sim(A\vee\sim A)$

$$\dfrac{\dfrac{[\sim(A\vee\sim A)]^4 \quad \dfrac{\dfrac{\dfrac{[\sim(A\vee\sim A)]^4 \quad \dfrac{[A]^2}{A\vee\sim A}\vee I_1 \quad [\bot]^1}{\bot}\supset E,1}{\sim A}\supset I,2}{A\vee\sim A}\vee I_2 \quad [\bot]^3}{\bot}\supset E,3}{\sim\sim(A\vee\sim A)}\supset I,4$$

3. Find normal derivations to last week's exercises 1 (b), 2 (c) and 3 (a).

$(A\supset(B\supset C))\supset((A\supset B)\supset(A\supset C))$

$$\dfrac{\dfrac{\dfrac{\dfrac{[A\supset(B\supset C)]^6 \quad [A]^4 \quad \dfrac{[B\supset C]^3 \quad \dfrac{[A\supset B]^5 \quad [A]^4 \quad [B]^1}{B}\supset E,1 \quad [C]^2}{C}\supset E,2}{C}\supset E,3}{A\supset C}\supset I,4}{(A\supset B)\supset(A\supset C)}\supset I,5}{(A\supset(B\supset C))\supset((A\supset B)\supset(A\supset C))}\supset I,6$$

$A\vee B\supset\sim(\sim A\,\&\,\sim B)$

$$\dfrac{\dfrac{\dfrac{[A\vee B]^7 \quad \dfrac{[\sim A\,\&\,\sim B]^6 \quad \dfrac{[\sim A]^3 \quad [A]^5 \quad [\bot]^1}{\bot}\supset E,1}{\bot}\&E,3 \quad \dfrac{[\sim A\,\&\,\sim B]^6 \quad \dfrac{[\sim B]^4 \quad [B]^5 \quad [\bot]^2}{\bot}\supset E,2}{\bot}\&E,4}{\bot}\vee E,5}{\sim(\sim A\,\&\,\sim B)}\supset I,6}{A\vee B\supset\sim(\sim A\,\&\,\sim B)}\supset I,7$$

$(A\,\&\,\sim A)\supset B$

$$\dfrac{\dfrac{[A\,\&\,\sim A]^3 \quad \dfrac{\dfrac{[\sim A]^2 \quad [A]^2 \quad [\bot]^1}{\bot}\supset E,1}{B}\bot E}{B}\&E,2}{(A\,\&\,\sim A)\supset B}\supset I,3$$

4. During the lecture it was shown how the special elimination rules can be obtained as special cases from the general elimination rules. Now derive the general elimination rules from the special elimination rules.

   Recall that the special elimination rules are the following:

$$\dfrac{A\,\&\,B}{A}\&E_1 \qquad \dfrac{A\,\&\,B}{B}\&E_2 \qquad \dfrac{A\supset B \quad A}{B}\supset E$$

   and the general elimination rules are the following:

$$\dfrac{A\,\&\,B \quad \begin{matrix}[A,B]\\ \vdots\\ C\end{matrix}}{C}\&E \qquad \dfrac{A\supset B \quad A \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\supset E.$$

We will start with the conjunction elimination. Assume that we have the premisses of the general conjunction elimination rule: $A \& B$ and the derivation $\begin{matrix} A,B \\ \vdots \\ C \end{matrix}$. Then we can derive the conclusion C using only the special elimination rule:

$$\begin{matrix} \dfrac{A \& B}{A}\,{\scriptstyle \&E_1} \quad \dfrac{A \& B}{B}\,{\scriptstyle \&E_2} \\ \vdots \\ C \end{matrix}$$

The case of implication proceeds in a similar fashion: assume $A \supset B$, A and the derivation $\begin{matrix} B \\ \vdots \\ C \end{matrix}$. Then we can derive C as follows:

$$\begin{matrix} \dfrac{A \supset B \quad A}{B}\,{\scriptstyle \supset E} \\ \vdots \\ C \end{matrix}$$

5. Use the sequent calculus rules presented during the lecture (the left and right rules on page 89 in the course book) to prove $\Rightarrow C$, where C is

   (a) $(A \supset B) \supset (\sim B \supset \sim A)$ (contraposition)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,{\scriptstyle Ax} \quad \dfrac{\dfrac{}{B \Rightarrow B}\,{\scriptstyle Ax} \quad \dfrac{}{\bot \Rightarrow \bot}\,{\scriptstyle L\bot}}{B, B \supset \bot \Rightarrow \bot}\,{\scriptstyle L\supset}}{A, B \supset \bot, A \supset B \Rightarrow \bot}\,{\scriptstyle L\supset}}{B \supset \bot, A \supset B \Rightarrow A \supset \bot}\,{\scriptstyle R\supset}}{A \supset B \Rightarrow (B \supset \bot) \supset (A \supset \bot)}\,{\scriptstyle R\supset}}{\Rightarrow (A \supset B) \supset ((B \supset \bot) \supset (A \supset \bot))}\,{\scriptstyle R\supset}$$

   (b) $\sim(A \& B) \supset\subset (A \supset \sim B)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,{\scriptstyle Ax} \quad \dfrac{}{B \Rightarrow B}\,{\scriptstyle Ax}}{B, A \Rightarrow A \& B}\,{\scriptstyle R\&} \quad \dfrac{}{\bot \Rightarrow \bot}\,{\scriptstyle L\bot}}{B, A, \sim(A \& B) \Rightarrow \bot}\,{\scriptstyle L\supset}}{A, \sim(A \& B) \Rightarrow \sim B}\,{\scriptstyle R\supset}}{\sim(A \& B) \Rightarrow A \supset \sim B}\,{\scriptstyle R\supset}}{\Rightarrow \sim(A \& B) \supset (A \supset \sim B)}\,{\scriptstyle R\supset} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,{\scriptstyle Ax} \quad \dfrac{\dfrac{}{B \Rightarrow B}\,{\scriptstyle Ax} \quad \dfrac{}{\bot \Rightarrow \bot}\,{\scriptstyle L\bot}}{B, \sim B \Rightarrow \bot}\,{\scriptstyle L\supset}}{A, B, A \supset \sim B \Rightarrow \bot}\,{\scriptstyle L\supset}}{A \& B, A \supset \sim B \Rightarrow \bot}\,{\scriptstyle L\&}}{A \supset \sim B \Rightarrow \sim(A \& B)}\,{\scriptstyle R\supset}}{\Rightarrow (A \supset \sim B) \supset \sim(A \& B)}\,{\scriptstyle R\supset}}{\Rightarrow \sim(A \& B) \supset (A \supset \sim B) \,\&\, (A \supset \sim B) \supset \sim(A \& B)}\,{\scriptstyle R\&}$$