Fte263 / 582418 Proof Theory, spring 2004. Solutions 3

1. Give sequent calculus proof in **G0ip** (this means **G0i** restricted to the propositional part, that is, the part without the quantifier rules) of $\Rightarrow C$, where C is

   (a) $(A \supset B) \supset (\sim B \supset \sim A)$ (contraposition)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,{\scriptstyle Ax} \quad \dfrac{\dfrac{}{B \Rightarrow B}\,{\scriptstyle Ax} \quad \dfrac{}{\bot \Rightarrow \bot}\,{\scriptstyle L\bot}}{B, \sim B \Rightarrow \bot}\,{\scriptstyle L\supset}}{A, \sim B, A \supset B \Rightarrow \bot}\,{\scriptstyle L\supset}}{\sim B, A \supset B \Rightarrow \sim A}\,{\scriptstyle R\supset}}{A \supset B \Rightarrow \sim B \supset \sim A}\,{\scriptstyle R\supset}}{\Rightarrow (A \supset B) \supset (\sim B \supset \sim A)}\,{\scriptstyle R\supset}$$

   (b) $A \vee B \supset \sim(\sim A \,\&\, \sim B)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,{\scriptstyle Ax} \quad \dfrac{\dfrac{}{\bot \Rightarrow \bot}\,{\scriptstyle L\bot}}{\bot, \sim B \Rightarrow \bot}\,{\scriptstyle Wk}}{\sim A, \sim B, A \Rightarrow \bot}\,{\scriptstyle L\supset}}{\sim A \,\&\, \sim B, A \Rightarrow \bot}\,{\scriptstyle L\&}}{A \Rightarrow \sim(\sim A \,\&\, \sim B)}\,{\scriptstyle R\supset} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,{\scriptstyle Ax}}{\sim A, B \Rightarrow B}\,{\scriptstyle Wk} \quad \dfrac{}{\bot \Rightarrow \bot}\,{\scriptstyle L\bot}}{\sim A, \sim B, B \Rightarrow \bot}\,{\scriptstyle L\supset}}{\sim A \,\&\, \sim B, B \Rightarrow \bot}\,{\scriptstyle L\&}}{B \Rightarrow \sim(\sim A \,\&\, \sim B)}\,{\scriptstyle R\supset}}{A \vee B \Rightarrow \sim(\sim A \,\&\, \sim B)}\,{\scriptstyle L\vee}}{\Rightarrow (A \vee B) \supset \sim(\sim A \,\&\, \sim B)}\,{\scriptstyle R\supset}$$

   (c) $\sim(A \supset B) \supset \sim\sim A \,\&\, \sim B$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,{\scriptstyle Ax} \quad \dfrac{}{\bot \Rightarrow B}\,{\scriptstyle L\bot}}{A, \sim A \Rightarrow B}\,{\scriptstyle L\supset}}{\sim A \Rightarrow A \supset B}\,{\scriptstyle R\supset} \quad \dfrac{}{\bot \Rightarrow \bot}\,{\scriptstyle L\bot}}{\sim A, \sim(A \supset B) \Rightarrow \bot}\,{\scriptstyle L\supset}}{\sim(A \supset B) \Rightarrow \sim\sim A}\,{\scriptstyle R\supset} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,{\scriptstyle Ax}}{B, A \Rightarrow B}\,{\scriptstyle Wk}}{B \Rightarrow A \supset B}\,{\scriptstyle R\supset} \quad \dfrac{}{\bot \supset \bot}\,{\scriptstyle L\bot}}{B, \sim(A \supset B) \Rightarrow \bot}\,{\scriptstyle L\supset}}{\sim(A \supset B) \Rightarrow \sim B}\,{\scriptstyle R\supset}}{\sim(A \supset B), \sim(A \supset B) \Rightarrow \sim\sim A \,\&\, \sim B}\,{\scriptstyle R\&}}{\sim(A \supset B) \Rightarrow \sim\sim A \,\&\, \sim B}\,{\scriptstyle Ctr}}{\Rightarrow \sim(A \supset B) \supset \sim\sim A \,\&\, \sim B}\,{\scriptstyle R\supset}$$

   (d) $\sim A \vee B \supset (A \supset B)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,{\scriptstyle Ax} \quad \dfrac{}{\bot \Rightarrow B}\,{\scriptstyle L\bot}}{A, \sim A \Rightarrow B}\,{\scriptstyle L\supset} \quad \dfrac{}{B \Rightarrow B}\,{\scriptstyle Ax}}{A, \sim A \vee B \Rightarrow B}\,{\scriptstyle L\vee}}{\sim A \vee B \Rightarrow A \supset B}\,{\scriptstyle R\supset}}{\Rightarrow \sim A \vee B \supset (A \supset B)}\,{\scriptstyle R\supset}$$

2. Prove that in **G0ip** each context-independent rule is interderivable with its context-sharing version:

$$\dfrac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \,\&\, B}\,{\scriptstyle R\&_{ind}} \qquad \dfrac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \,\&\, B}\,{\scriptstyle R\&_{sh}}$$

$$\dfrac{\Gamma, A \Rightarrow C \quad \Delta, B \Rightarrow C}{\Gamma, \Delta, A \vee B \Rightarrow C}\,{\scriptstyle L\vee_{ind}} \qquad \dfrac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}\,{\scriptstyle L\vee_{sh}}$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, B \Rightarrow C}{\Gamma, \Delta, A \supset B \Rightarrow C}\; L{\supset}_{ind} \qquad \frac{\Gamma \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C}\; L{\supset}_{sh}$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, A \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\; Cut_{ind} \qquad \frac{\Gamma \Rightarrow A \quad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow C}\; Cut_{sh}$$

We will start with right conjunction: We will assume the premisses of $R\&_{ind}$, use weakening repeatedly to get the same context to both sides and derive the conclusion of $R\&_{ind}$ using $R\&_{sh}$ as follows (Wk^* means Wk applied $0 \ldots n$ times):

$$\frac{\dfrac{\Gamma \Rightarrow A}{\Gamma, \Delta \Rightarrow A}\; Wk^* \qquad \dfrac{\Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow B}\; Wk^*}{\Gamma, \Delta \Rightarrow A \,\&\, B}\; R\&_{sh}$$

The other direction can be proved by assuming the premisses of $R\&_{sh}$, by applying $R\&_{ind}$ and by using contraction repeatedly in order to remove duplicated formulae from the context:

$$\frac{\dfrac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma, \Gamma \Rightarrow A \,\&\, B}\; R\&_{ind}}{\Gamma \Rightarrow A \,\&\, B}\; Ctr^*$$

The other cases are similar:

$$\frac{\dfrac{\Gamma, A \Rightarrow C}{\Gamma, \Delta, A \Rightarrow C}\; Wk^* \qquad \dfrac{\Delta, B \Rightarrow C}{\Gamma, \Delta, B \Rightarrow C}\; Wk^*}{\Gamma, \Delta, A \vee B \Rightarrow C}\; L{\vee}_{sh} \qquad \frac{\dfrac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, \Gamma, A \vee B \Rightarrow C}\; L{\vee}_{ind}}{\Gamma, A \vee B \Rightarrow C}\; Ctr^*$$

$$\frac{\dfrac{\Gamma \Rightarrow A}{\Gamma, \Delta \Rightarrow A}\; Wk^* \qquad \dfrac{\Delta, B \Rightarrow C}{\Gamma, \Delta, B \Rightarrow C}\; Wk^*}{\Gamma, \Delta, A \supset B \Rightarrow C}\; L{\supset}_{sh} \qquad \frac{\dfrac{\Gamma \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, \Gamma, A \supset B \Rightarrow C}\; L{\supset}_{ind}}{\Gamma, A \supset B \Rightarrow C}\; Ctr^*$$

$$\frac{\dfrac{\Gamma \Rightarrow A}{\Gamma, \Delta \Rightarrow A}\; Wk^* \qquad \dfrac{\Delta, A \Rightarrow C}{\Gamma, \Delta, A \Rightarrow C}\; Wk^*}{\Gamma, \Delta \Rightarrow C}\; Cut_{sh} \qquad \frac{\dfrac{\Gamma \Rightarrow A \quad \Gamma, A \Rightarrow C}{\Gamma, \Gamma \Rightarrow C}\; Cut_{ind}}{\Gamma \Rightarrow C}\; Ctr^*$$

□

3. Show that reductio ad absurdum (*Raa*) is derivable using the natural deduction rules for intuitionistic logic and the rule of excluded middle (*Em*).

   The proof proceeds by assuming the premisses of the rule *Raa* and deriving its conlusion without using the rule. The rule is $\dfrac{\begin{matrix}[\sim A] \\ \vdots \\ \bot\end{matrix}}{A}$ so we will assume that we have the derivation $\begin{matrix}\sim A \\ \vdots \\ \bot\end{matrix}$ and derive A as follows:

$$\frac{[A]^1 \qquad \dfrac{\begin{matrix}[\sim A]^1 \\ \vdots \\ \bot\end{matrix}}{A}\; \bot E}{A}\; Em{,}1$$

□

4. Prove in natural deduction for classical logic (intuitionistic logic + *Em*):

(a) $\sim(\sim A \mathbin{\&} \sim B) \supset A \lor B$

$$\dfrac{\dfrac{[B]^3 \qquad \dfrac{\dfrac{[A]^2}{A \lor B}\;{\scriptstyle \lor I} \qquad \dfrac{\dfrac{[\sim(\sim A \mathbin{\&} \sim B)] \qquad \dfrac{[\sim A]^2 \quad [\sim B]^3}{\sim A \mathbin{\&} \sim B}\;{\scriptstyle \& I} \qquad [\bot]^1}{\bot}\;{\scriptstyle \supset E,1}}{A \lor B}\;{\scriptstyle \bot E}}{A \lor B}\;{\scriptstyle Em,2}}{A \lor B}\;{\scriptstyle Em,3}}{\sim(\sim A \mathbin{\&} \sim B) \supset A \lor B}\;{\scriptstyle \supset I,4}$$

(b) $(A \supset B) \supset\subset (\sim A \lor B)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \supset B]^2 \quad [A]^3 \quad [B]^1}{B}\;{\scriptstyle \supset E,1}}{\sim A \lor B}\;{\scriptstyle \lor I}}{(A \supset B) \supset (\sim A \lor B)}\;{\scriptstyle \supset I,2} \qquad \dfrac{\dfrac{[\sim A]^3}{\sim A \lor B}\;{\scriptstyle \lor I}}{(A \supset B) \supset (\sim A \lor B)}\;{\scriptstyle \supset I}}{(A \supset B) \supset (\sim A \lor B)}\;{\scriptstyle Em,3} \qquad \dfrac{\dfrac{[\sim A \lor B]^7 \qquad \dfrac{\dfrac{[\sim A]^6 \quad [A]^5 \quad \dfrac{[\bot]^4}{B}\;{\scriptstyle \bot E}}{B}\;{\scriptstyle \supset E,4}}{A \supset B}\;{\scriptstyle \supset I,5} \qquad \dfrac{[B]^6}{A \supset B}\;{\scriptstyle \supset I}}{A \supset B}\;{\scriptstyle \lor E,6}}{(\sim A \lor B) \supset (A \supset B)}\;{\scriptstyle \supset I,4}}{(A \supset B) \supset\subset (\sim A \lor B)}\;{\scriptstyle \& I}$$

(c) $((A \supset B) \supset A) \supset A$ (Peirce's law)

$$\dfrac{\dfrac{[A]^5}{((A \supset B) \supset A) \supset A}\;{\scriptstyle \supset I} \qquad \dfrac{\dfrac{[(A \supset B) \supset A]^4 \qquad \dfrac{\dfrac{[\sim A]^6 \quad [A]^2 \quad \dfrac{[\bot]^1}{B}\;{\scriptstyle \bot E}}{B}\;{\scriptstyle \supset E,1}}{A \supset B}\;{\scriptstyle \supset I,2} \qquad [A]^3}{A}\;{\scriptstyle \supset E,3}}{((A \supset B) \supset A) \supset A}\;{\scriptstyle \supset I,4}}{((A \supset B) \supset A) \supset A}\;{\scriptstyle Em,5,6}$$

(d) $(A \supset B \lor C) \supset (A \supset B) \lor (A \supset C)$ (disjunction property under hypothesis)

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \supset B \lor C]^6 \quad [A]^5 \quad [B \lor C]^1}{B \lor C}\;{\scriptstyle \supset E,1} \qquad \dfrac{\dfrac{[B]^2}{A \supset B}\;{\scriptstyle \supset I}}{(A \supset B) \lor (A \supset C)}\;{\scriptstyle \lor I} \qquad \dfrac{\dfrac{[C]^2}{A \supset C}\;{\scriptstyle \supset I}}{(A \supset B) \lor (A \supset C)}\;{\scriptstyle \lor I}}{(A \supset B) \lor (A \supset C)}\;{\scriptstyle \lor E,2} \qquad \dfrac{\dfrac{\dfrac{[A]^4 \quad [\sim A]^5 \quad \dfrac{[\bot]^3}{C}\;{\scriptstyle \bot E}}{C}\;{\scriptstyle \supset E,3}}{A \supset C}\;{\scriptstyle \supset I,4}}{(A \supset B) \lor (A \supset C)}\;{\scriptstyle \lor I}}{(A \supset B) \lor (A \supset C)}\;{\scriptstyle Em,5}}{(A \supset B \lor C) \supset (A \supset B) \lor (A \supset C)}\;{\scriptstyle \supset I,6}$$

5. Give sequent calculus proof in **G3cp** of $\Rightarrow D$, where D is

(a) $\sim(\sim A \mathbin{\&} \sim B) \supset A \lor B$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A, B, \bot}\;{\scriptstyle Ax}}{\Rightarrow A, B, \sim A}\;{\scriptstyle R\supset} \qquad \dfrac{\dfrac{}{B \Rightarrow A, B, \bot}\;{\scriptstyle Ax}}{\Rightarrow A, B, \sim B}\;{\scriptstyle R\supset}}{\Rightarrow A, B, \sim A \mathbin{\&} \sim B}\;{\scriptstyle R\&} \qquad \dfrac{}{\bot \Rightarrow A, B}\;{\scriptstyle L\bot}}{\sim(\sim A \mathbin{\&} \sim B) \Rightarrow A, B}\;{\scriptstyle L\supset}}{\sim(\sim A \mathbin{\&} \sim B) \Rightarrow A \lor B}\;{\scriptstyle R\lor}}{\Rightarrow \sim(\sim A \mathbin{\&} \sim B) \supset A \lor B}\;{\scriptstyle R\supset}$$

(b) $(A \supset B) \supset\subset (\sim A \vee B)$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A \Rightarrow \bot, B, A}\,{\scriptstyle Ax}
}{\Rightarrow \sim A, B, A}\,{\scriptstyle R\supset}
\quad
\dfrac{}{B \Rightarrow \sim A, B}\,{\scriptstyle Ax}
}{A \supset B \Rightarrow \sim A, B}\,{\scriptstyle L\supset}
}{A \supset B \Rightarrow \sim A \vee B}\,{\scriptstyle R\vee}
}{\Rightarrow (A \supset B) \supset (\sim A \vee B)}\,{\scriptstyle R\supset}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A \Rightarrow B, A}\,{\scriptstyle Ax}
\quad
\dfrac{}{B, A \Rightarrow B}\,{\scriptstyle Ax}
}{A, \sim A \Rightarrow B}\,{\scriptstyle L\supset}
\quad
\dfrac{}{A, B \Rightarrow B}\,{\scriptstyle Ax}
}{A, \sim A \vee B \Rightarrow B}\,{\scriptstyle L\vee}
}{\sim A \vee B \Rightarrow A \supset B}\,{\scriptstyle R\supset}
}{\Rightarrow (\sim A \vee B) \supset (A \supset B)}\,{\scriptstyle R\supset}
}{\Rightarrow (A \supset B) \supset\subset (\sim A \vee B)}\,{\scriptstyle R\&}
$$

(c) $((A \supset B) \supset A) \supset A$ (Peirce's law)

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A \Rightarrow A, B}\,{\scriptstyle Ax}
}{\Rightarrow A, A \supset B}\,{\scriptstyle R\supset}
\quad
\dfrac{}{A \Rightarrow A}\,{\scriptstyle Ax}
}{(A \supset B) \supset A \Rightarrow A}\,{\scriptstyle L\supset}
}{\Rightarrow ((A \supset B) \supset A) \supset A}\,{\scriptstyle R\supset}
$$

(d) $(A \supset B \vee C) \supset (A \supset B) \vee (A \supset C)$ (disjunction property under hypothesis)

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A, A \Rightarrow A}\,{\scriptstyle Ax}
\quad
\dfrac{
\dfrac{}{A, A, B \Rightarrow B, C}\,{\scriptstyle Ax}
\quad
\dfrac{}{A, A, C \Rightarrow B, C}\,{\scriptstyle Ax}
}{A, A, B \vee C \Rightarrow B, C}\,{\scriptstyle L\vee}
}{A, A, A \supset B \vee C \Rightarrow B, C}\,{\scriptstyle L\supset}
}{A, A \supset B \vee C \Rightarrow B, A \supset C}\,{\scriptstyle R\supset}
}{A \supset B \vee C \Rightarrow A \supset B, A \supset C}\,{\scriptstyle R\supset}
}{(A \supset B \vee C) \Rightarrow (A \supset B) \vee (A \supset C)}\,{\scriptstyle R\vee}
}{\Rightarrow (A \supset B \vee C) \supset (A \supset B) \vee (A \supset C)}\,{\scriptstyle R\supset}
$$