Fte263 / 582418 Proof Theory, spring 2004. Solutions 4

1. Show that in **G0ip** without weakening $A \supset (B \supset A)$ is not derivable. Show that in **G0ip** without contraction $(A \supset (A \supset B)) \supset (A \supset B)$ is not derivable.

   (a) **Proof:** We will go through all possible derivations starting from the conclusion and show that there are always some branches of the derivation tree that are not axioms or instances of $L\bot$.

   Starting from $\Rightarrow A \supset (B \supset A)$, we will see that the last step must be $R\supset$:

$$\frac{A \Rightarrow B \supset A}{\Rightarrow A \supset (B \supset A)}\, R\supset .$$

   Then we can use either contraction or right implication:

   i. *Ctr*: $\dfrac{A, A \Rightarrow B \supset A}{A \supset (B \supset A)}\, Ctr$ Then we can continue with another *Ctr* or $R\supset$:

      A. $\dfrac{A, A, A \Rightarrow B \supset A}{A, A \Rightarrow B \supset A}\, Ctr$

      B. $\dfrac{A, A, B \Rightarrow A}{A, A \Rightarrow B \supset A}\, R\supset$ From now on, we can only apply contraction, but that will never lead to an axiom.

   ii. $R\supset$: $\dfrac{A, B \Rightarrow A}{A \Rightarrow B \supset A}\, R\supset$ Also this branch of the proof search fails because we can only use contraction.

□

   (b) $(A \supset (A \supset B)) \supset (A \supset B)$ Only $R\supset$ is applicable:

$$\frac{A \supset (A \supset B) \Rightarrow A \supset B}{(A \supset (A \supset B)) \supset (A \supset B)}\, R\supset$$

   Here we have three choices: right implication, left implication or weakening:

   i. $\dfrac{A, A \supset (A \supset B) \Rightarrow B}{A \supset (A \supset B) \Rightarrow A \supset B}\, R\supset$ Now we can apply either weakening (two cases) or left implication:

      A. (Weakening applied to A) $\dfrac{A \supset (A \supset B) \Rightarrow B}{A, A \supset (A \supset B) \Rightarrow B}\, Wk$ Again, we can apply either weakening or left implication:

      - $\dfrac{\Rightarrow B}{A \supset (A \supset B) \Rightarrow B}\, Wk$ Here we cannot continue.
      - $\dfrac{\Rightarrow A \quad A \supset B \Rightarrow B}{A \supset (A \supset B) \Rightarrow B}\, L\supset$ We cannot proceed with the first premiss $\Rightarrow A$ so there is no need to continue with the second one.

      B. (Weakening applied to $A \supset (A \supset B)$) $\dfrac{\dfrac{\Rightarrow B}{A \Rightarrow B}\, Wk}{A, A \supset (A \supset B) \Rightarrow B}\, Wk$ Cannot continue.

      C. (Left implication with A in the first premiss):

$$\frac{\overline{A \Rightarrow A}\, Ax \quad \dfrac{\Rightarrow A \quad \overline{B \Rightarrow B}\, Ax}{A \supset B \Rightarrow B}\, L\supset}{A, A \supset (A \supset B) \Rightarrow B}\, L\supset$$

      Cannot continue from $\Rightarrow A$.

D. (Left implication with A in the second presmiss): $\dfrac{\Rightarrow A \quad A, A\supset B \Rightarrow B}{A, A\supset(A\supset B)\Rightarrow B}\,L\supset$

Cannot contine from $\Rightarrow A$.

ii. $\dfrac{}{A\supset(A\supset B)\Rightarrow A\supset B}\,L\supset$

iii. $\dfrac{}{A\supset(A\supset B)\Rightarrow A\supset B}\,Wk$

2. Show that generalized axioms $A,\Gamma\Rightarrow\Delta,A$ are derivable in **G3cp**.

   Proof: By induction on the formula structure. Base case: A is an atomic formula (P) or A is falsity ($\bot$). If $A=P$, then $A,\Gamma\Rightarrow\Delta,A$ is equal to $P,\Gamma\Rightarrow\Delta,P$ which is an axiom. If $A=\bot$, then we have $\bot,\Gamma\Rightarrow\Delta,\bot$, which is an instance of $L\bot$.

   Inductive hypothesis: Suppose that $B,\Gamma\Rightarrow\Delta,B$ and $C,\Gamma\Rightarrow\Delta,C$ are derivable. This hypothesis will be denoted with *IH* below.

   Induction step: We will have to show that $A,\Gamma\Rightarrow\Delta,A$ is derivable. We have the following cases: $A=B\vee C$, $A=B\,\&C$ and $A=B\supset C$. We will start with $B\vee C$:

$$\dfrac{\dfrac{\dfrac{}{B,\Gamma\Rightarrow\Delta,B,C}\,IH \quad \dfrac{}{C,\Gamma\Rightarrow\Delta,B,C}\,IH}{B\vee C,\Gamma\Rightarrow\Delta,B,C}\,L\vee}{B\vee C,\Gamma\Rightarrow\Delta,B\vee C}\,R\vee$$

   Premisses $B,\Gamma\Rightarrow\Delta,B,C$ and $C,\Gamma\Rightarrow\Delta,B,C$ are derivable by the inductive hypothesis. The other cases are similar:

$$\dfrac{\dfrac{\dfrac{}{B,C,\Gamma\Rightarrow\Delta,B}\,IH \quad \dfrac{}{B,C,\Gamma\Rightarrow\Delta,C}\,IH}{B,C,\Gamma\Rightarrow\Delta,B\,\&C}\,R\&}{B\,\&C,\Gamma\Rightarrow\Delta,B\,\&C}\,L\& \qquad \dfrac{\dfrac{\dfrac{}{B,\Gamma\Rightarrow B}\,IH \quad \dfrac{}{C,B,\Gamma\Rightarrow\Delta,C}\,IH}{B,B\supset C,\Gamma\Rightarrow\Delta,C}\,L\supset}{B\supset C,\Gamma\Rightarrow\Delta,B\supset C}\,R\supset$$

   □

3. Using the calculus **G3cp** find conjunctive normal form for the following formulae

   (a) $(A\,\&B)\supset(A\supset(B\,\&\sim A))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A,B,A\Rightarrow B}\,Ax \quad \dfrac{A,B,A,A\Rightarrow\bot}{A,B,A\Rightarrow A\supset\bot}\,R\supset}{A,B,A\Rightarrow B\&(A\supset\bot)}\,R\&}{A,B\Rightarrow A\supset(B\&(A\supset\bot))}\,R\supset}{A\&B\Rightarrow A\supset(B\&(A\supset\bot))}\,L\&}{\Rightarrow(A\&B)\supset(A\supset(B\&(A\supset\bot)))}\,R\supset$$

   We get one topsequent ($A,B,A,A\Rightarrow\bot$) which is not an axiom or conclusion of L$\bot$. We delete the repetitions of A and get the regular sequent $A,B\Rightarrow\bot$. According to Definition 3.1.3 (page 51), this corresponds to the trace formula $\sim(A\,\&B)$ which is classically equivalent with the CNF formula $\sim A\vee\sim B$.

   (b) $(A\vee(\sim B\,\&B))\ \&\ \sim(B\,\&\sim C)$

$$\dfrac{\dfrac{\dfrac{\dfrac{B\Rightarrow A,\bot}{\Rightarrow A,B\supset\bot}\,R\supset \quad \Rightarrow A,B}{\Rightarrow A,(B\supset\bot)\&B}\,R\&}{\Rightarrow A\vee((B\supset\bot)\&B)}\,R\vee \quad \dfrac{\dfrac{\dfrac{B\Rightarrow\bot,C \quad \dfrac{}{B,\bot\Rightarrow\bot}\,L\bot}{B,C\supset\bot\Rightarrow\bot}\,L\supset}{B\&(C\supset\bot)\Rightarrow\bot}\,L\&}{\Rightarrow(B\&(C\supset\bot))\supset\bot}\,R\supset}{\Rightarrow(A\vee((B\supset\bot)\&B))\&((B\&(C\supset\bot))\supset\bot)}\,R\&$$

This time we get three topsequents: $B \Rightarrow A, \bot$, $\Rightarrow A, B$ and $B \Rightarrow \bot, C$. Their trace formulae are $B \supset A$, $A \vee B$ and $B \supset C$. Thus, the original formula is equivalent with their conjunction: $(B \supset A)\,\&\,(A \vee B)\,\&\,(B \supset C)$ As explained on page 52, $B \supset A$ is classically equivalent with $\sim B \vee A$ (and $B \supset C$ with $\sim B \vee C$) so we get the CNF formula $(\sim B \vee A)\,\&\,(A \vee B)\,\&\,(\sim B \vee C)$.

(c) $(A \vee \sim\sim B) \supset (\sim B \supset A)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A,B}\,Ax \quad \dfrac{\dfrac{\dfrac{}{B \Rightarrow A,B,\bot}\,Ax}{\Rightarrow A,B,B \supset \bot}\,R\supset \quad \dfrac{}{\bot \Rightarrow A,B}\,L\bot}{(B \supset \bot) \supset \bot \Rightarrow A,B}\,L\supset}{A \vee ((B \supset \bot) \supset \bot) \Rightarrow A,B}\,L\vee \quad \dfrac{\dfrac{}{A,\bot \Rightarrow A}\,Ax \quad \dfrac{\dfrac{\dfrac{}{\bot,B \Rightarrow A,\bot}\,L\bot}{\bot \Rightarrow A,B \supset \bot}\,R\supset \quad \dfrac{}{\bot,\bot \Rightarrow A}\,L\bot}{(B \supset \bot) \supset \bot,\bot \Rightarrow A}\,L\supset}{A \vee ((B \supset \bot) \supset \bot),\bot \Rightarrow A}\,L\vee}{A \vee ((B \supset \bot) \supset \bot), B \supset \bot \Rightarrow A}\,L\supset}{A \vee ((B \supset \bot) \supset \bot) \Rightarrow (B \supset \bot) \supset A}\,R\supset}{\Rightarrow (A \vee ((B \supset \bot) \supset \bot)) \supset ((B \supset \bot) \supset A)}\,R\supset
$$

All the branches terminate so the formula is a theorem and we have an empty conjunction, which corresponds to $\top$.

4. Complete the proof of height-preserving contraction for **G3cp** (Theorem 3.2.2 on page 53 of the book) presented in the last lecture.

The base case of the induction was handled during the lecture together with the case where the contraction formula is not principal. If the contraction formula is principal, there are six subcases according to the last rule applied before the contraction. Of these, $L\,\&$ and $R \supset$ were shown and $R\,\&$, $R\vee$ and $L \supset$ can be found in the book so the case where the rule is $L\vee$ remains:

We have to show that if $\Gamma, B \vee C, B \vee C \Rightarrow \Delta$ is derivable in $n+1$ steps, then also $\Gamma, B \vee C \Rightarrow \Delta$ is derivable in $n+1$ steps. We have the inductive hypothesis that for all formulae A if $\Gamma, A, A \Rightarrow \Delta$ is derivable in n steps then also $\Gamma, A \Rightarrow \Delta$ is derivable in n steps and if $\Gamma \Rightarrow \Delta, A, A$ is derivable in n steps then also $\Gamma \Rightarrow \Delta, A$ is derivable in n steps.

The last rule is $L\vee$ so we have

$$
\dfrac{\Gamma, B \vee C, B \Rightarrow \Delta \quad \Gamma, B \vee C, C \Rightarrow \Delta}{\Gamma, B \vee C, B \vee C \Rightarrow \Delta}\,L\vee
$$

So the premisses of this rule are derivable in n steps. Then by invertibility of the rule $L\vee$, also $\Gamma, B, B \Rightarrow \Delta$ and $\Gamma, C, C \Rightarrow \Delta$ are also derivable in n steps. Then by the inductive hypothesis, we have a derviation:

$$
\dfrac{\dfrac{\vdash_n \Gamma, B, B \Rightarrow \Delta}{\vdash_n \Gamma, B \Rightarrow \Delta}\,IH \quad \dfrac{\vdash_n \Gamma, C, C \Rightarrow \Delta}{\vdash_n \Gamma, C \Rightarrow \Delta}\,IH}{\vdash_{n+1} \Gamma, B \vee C \Rightarrow \Delta}\,L\vee
$$

Thus $\Gamma, B \vee C \Rightarrow \Delta$ is derivable in $n+1$ steps. □