Fte263 / 582418 Proof Theory, spring 2004. Solutions 5

2. Use a proof editor to give sequent calculus proof in **G3cp** of $\Rightarrow D$, where D is

   (a) $(A \supset B \lor C) \supset (A \supset B) \lor (A \supset C)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A,A \Rightarrow B,C,A}\,Ax \quad \dfrac{\dfrac{}{B,A,A \Rightarrow B,C}\,Ax \quad \dfrac{}{C,A,A \Rightarrow B,C}\,Ax}{B \lor C,A,A \Rightarrow B,C}\,L\lor}{A \supset (B \lor C),A,A \Rightarrow B,C}\,L\supset}{A \supset (B \lor C),A \Rightarrow B,A \supset C}\,R\supset}{A \supset (B \lor C) \Rightarrow A \supset B,A \supset C}\,R\supset}{A \supset (B \lor C) \Rightarrow (A \supset B) \lor (A \supset C)}\,R\lor}{\Rightarrow (A \supset (B \lor C)) \supset ((A \supset B) \lor (A \supset C))}\,R\supset$$

   (b) $((A \,\&\, \sim B) \,\&\, ((((A \lor (B \supset B)) \supset \sim(A \,\&\, B)) \lor \sim B) \supset \sim A)) \supset C$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A,B \Rightarrow C,B,(A \lor (B \supset B)) \supset ((A\&B) \supset \bot),\bot}\,Ax}{A \Rightarrow C,B,(A \lor (B \supset B)) \supset ((A\&B) \supset \bot),B \supset \bot}\,R\supset}{A \Rightarrow C,B,((A \lor (B \supset B)) \supset ((A\&B) \supset \bot)) \lor (B \supset \bot)}\,R\lor \quad \dfrac{\dfrac{}{A \Rightarrow C,B,A}\,Ax \quad \dfrac{}{A,\bot \Rightarrow C,B}\,L\bot}{A,A \supset \bot \Rightarrow C,B}\,L\supset}{A,(((A \lor (B \supset B)) \supset ((A\&B) \supset \bot)) \lor (B \supset \bot)) \supset (A \supset \bot) \Rightarrow C,B}\,L\supset \quad \dfrac{}{A,\bot,(((A \lor (B \supset B)) \supset ((A\&B) \supset \bot)) \lor (B \supset \bot)) \supset (A \supset \bot) \Rightarrow C}\,L\bot}{A,B \supset \bot,(((A \lor (B \supset B)) \supset ((A\&B) \supset \bot)) \lor (B \supset \bot)) \supset (A \supset \bot) \Rightarrow C}\,L\supset}{A\&(B \supset \bot),(((A \lor (B \supset B)) \supset ((A\&B) \supset \bot)) \lor (B \supset \bot)) \supset (A \supset \bot) \Rightarrow C}\,L\&}{(A\&(B \supset \bot))\&((((A \lor (B \supset B)) \supset ((A\&B) \supset \bot)) \lor (B \supset \bot)) \supset (A \supset \bot)) \Rightarrow C}\,L\&}{\Rightarrow ((A\&(B \supset \bot))\&((((A \lor (B \supset B)) \supset ((A\&B) \supset \bot)) \lor (B \supset \bot)) \supset (A \supset \bot))) \supset C}\,R\supset$$

   (c) $\sim((((A \supset A) \supset A) \,\&\, ((\sim A \supset A) \supset \sim A)) \lor (((B \supset B) \supset B) \,\&\, \sim B) \lor \sim\sim(\sim(A \supset B) \,\&\, \sim(B \supset \sim\sim A)))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{ax}{A,(\sim A \supset A) \supset \sim A \Rightarrow \bot,A}}{(\sim A \supset A) \supset \sim A \Rightarrow \bot,A \supset A}\,R\supset \quad \dfrac{\dfrac{\dfrac{\dfrac{ax}{A \Rightarrow \bot,A,A} \quad \dfrac{L\bot}{\bot,A \Rightarrow \bot,A}}{\sim A,A \Rightarrow \bot,A}\,L\supset}{A \Rightarrow \bot,\sim A \supset A}\,R\supset \quad \dfrac{\dfrac{ax}{A \Rightarrow \bot,A} \quad \dfrac{L\bot}{\bot,A \Rightarrow \bot}}{\sim A,A \Rightarrow \bot}\,L\supset}{(\sim A \supset A) \supset \sim A,A \Rightarrow \bot}\,L\supset}{(A \supset A) \supset A,(\sim A \supset A) \supset \sim A \Rightarrow \bot}\,L\supset}{((A \supset A) \supset A)\&((\sim A \supset A) \supset \sim A) \Rightarrow \bot}\,L\& \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{ax}{B,\sim B \Rightarrow \bot,B}}{\sim B \Rightarrow \bot,B \supset B}\,R\supset \quad \dfrac{\dfrac{ax}{B \Rightarrow \bot,B} \quad \dfrac{L\bot}{\bot,B \Rightarrow \bot}}{\sim B,B \Rightarrow \bot}\,L\supset}{(B \supset B) \supset B,\sim B \Rightarrow \bot}\,L\supset}{((B \supset B) \supset B)\&\sim B \Rightarrow \bot}\,L\& \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{ax}{B,A \Rightarrow \bot,\bot,\sim\sim A,B}}{A \Rightarrow \bot,\bot,B,B \supset \sim\sim A}\,R\supset \quad \dfrac{L\bot}{\bot,A \Rightarrow \bot,\bot,B}}{\sim(B \supset \sim\sim A),A \Rightarrow \bot,\bot,B}\,L\supset}{\sim(B \supset \sim\sim A) \Rightarrow \bot,\bot,A \supset B}\,R\supset \quad \dfrac{L\bot}{\bot,\sim(B \supset \sim\sim A) \Rightarrow \bot,\bot}}{\sim(A \supset B),\sim(B \supset \sim\sim A) \Rightarrow \bot,\bot}\,L\supset}{\sim(A \supset B)\&\sim(B \supset \sim\sim A) \Rightarrow \bot,\bot}\,L\&}{\Rightarrow \bot,\sim(\sim(A \supset B)\&\sim(B \supset \sim\sim A))}\,R\supset \quad \dfrac{L\bot}{\bot \Rightarrow \bot}}{\sim\sim(\sim(A \supset B)\&\sim(B \supset \sim\sim A)) \Rightarrow \bot}\,L\supset}{(((B \supset B) \supset B)\&\sim B) \lor \sim\sim(\sim(A \supset B)\&\sim(B \supset \sim\sim A)) \Rightarrow \bot}\,Lv}{(((A \supset A) \supset A)\&((\sim A \supset A) \supset \sim A)) \lor ((((B \supset B) \supset B)\&\sim B) \lor \sim\sim(\sim(A \supset B)\&\sim(B \supset \sim\sim A))) \Rightarrow \bot}\,Lv}{\Rightarrow \sim((((A \supset A) \supset A)\&((\sim A \supset A) \supset \sim A)) \lor ((((B \supset B) \supset B)\&\sim B) \lor \sim\sim(\sim(A \supset B)\&\sim(B \supset \sim\sim A))))}\,R\supset$$

3. Complete the proof of the admissibility of cut (Theorem 3.2.3, page 54) in the following cases:

   (a) The right premiss is an axiom or conclusion of L$\bot$.

   The remaining case is 2.4: $D = \bot$ and the first premiss $\Gamma \Rightarrow \Delta, \bot$ has been derived. There are six cases according to the last rule used in the derivation:

   - Rule$=$ $L\,\&$: $\Gamma = A \,\&\, B, \Gamma''$. The derivation

$$\dfrac{\dfrac{A,B,\Gamma'' \Rightarrow \Delta,\bot}{A \,\&\, B,\Gamma'' \Rightarrow \Delta,\bot}\,L\& \quad \dfrac{}{\bot,\Gamma' \Rightarrow \Delta'}\,L\bot}{A \,\&\, B,\Gamma'',\Gamma' \Rightarrow \Delta,\Delta'}\,Cut$$

is transformed into

$$\dfrac{\dfrac{A,B,\Gamma''\Rightarrow\Delta,\bot \qquad \dfrac{}{\bot,\Gamma'\Rightarrow\Delta'}\;L\bot}{A,B,\Gamma'',\Gamma'\Rightarrow\Delta,\Delta'}\;\mathit{Cut}}{A\,\&\,B,\Gamma'',\Gamma\Rightarrow\Delta,\Delta'}\;L\&$$

with a cut with lower cut height. (Suppose that the height of the sequent $A,B,\Gamma''\Rightarrow\Delta,\bot$ is n in the first derivation. Then the cut-height of the cut rule is $n+1+1=n+2$. In the second derivation, the cut-height is $n+1$.)

- Rule$=L\vee:\Gamma=A\vee B,\Gamma''$. The derivation

$$\dfrac{\dfrac{A,\Gamma''\Rightarrow\Delta,\bot \qquad B,\Gamma''\Rightarrow\Delta,\bot}{A\vee B,\Gamma''\Rightarrow\Delta,\bot}\;L\vee \qquad \dfrac{}{\bot,\Gamma'\Rightarrow\Delta'}\;L\bot}{A\vee B,\Gamma'',\Gamma'\Rightarrow\Delta,\Delta'}\;\mathit{Cut}$$

is transformed into

$$\dfrac{\dfrac{A,\Gamma''\Rightarrow\Delta,\bot \qquad \dfrac{}{\bot,\Gamma'\Rightarrow\Delta'}\;L\bot}{A,\Gamma'',\Gamma'\Rightarrow\Delta,\Delta'}\;\mathit{Cut} \qquad \dfrac{B,\Gamma''\Rightarrow\Delta,\bot \qquad \dfrac{}{\bot,\Gamma'\Rightarrow\Delta'}\;L\bot}{B,\Gamma'',\Gamma'\Rightarrow\Delta,\Delta'}\;\mathit{Cut}}{A\vee B,\Gamma'',\Gamma'\Rightarrow\Delta,\Delta'}\;L\vee$$

with two cuts with lower cut-height.

- Rule$=L\supset:\Gamma=A\supset B,\Gamma''$. The derivation

$$\dfrac{\dfrac{\Gamma''\Rightarrow\Delta,\bot,A \qquad B,\Gamma''\Rightarrow\Delta,\bot}{A\supset B,\Gamma''\Rightarrow\Delta,\bot}\;L\supset \qquad \dfrac{}{\bot,\Gamma'\Rightarrow\Delta'}\;L\bot}{A\supset B,\Gamma'',\Gamma'\Rightarrow\Delta,\Delta''}\;\mathit{Cut}$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\Gamma''\Rightarrow\Delta,\bot,A \qquad \dfrac{}{\bot,\Gamma'\Rightarrow\Delta'}\;L\bot}{\Gamma'',\Gamma'\Rightarrow\Delta,A,\Delta'}\;\mathit{Cut} \qquad \dfrac{B,\Gamma''\Rightarrow\Delta,\bot \qquad \dfrac{}{\bot,\Gamma'\Rightarrow\Delta'}\;L\bot}{B,\Gamma'',\Gamma'\Rightarrow\Delta,\Delta'}\;\mathit{Cut}}{A\supset B,\Gamma'',\Gamma',\Gamma''\Rightarrow\Delta,\Delta'}\;L\supset}{A\supset B,\Gamma'',\Gamma'\Rightarrow\Delta,\Delta'}\;\mathit{Ctr}$$

with two cuts with lower cut-height.

- Rule$=R\,\&:\Delta=\Delta'',A\,\&\,B$. The derivation

$$\dfrac{\dfrac{\Gamma\Rightarrow\Delta'',A,\bot \qquad \Gamma\Rightarrow\Delta'',B,\bot}{\Gamma\Rightarrow\Delta'',A\,\&\,B,\bot}\;R\& \qquad \dfrac{}{\bot,\Gamma'\Rightarrow\Delta'}\;L\bot}{\Gamma,\Gamma'\Rightarrow\Delta'',A\,\&\,B,\Delta'}\;\mathit{Cut}$$

is transformed into

$$\dfrac{\dfrac{\Gamma\Rightarrow\Delta'',A,\bot \qquad \dfrac{}{\bot,\Gamma'\Rightarrow\Delta'}\;L\bot}{\Gamma,\Gamma'\Rightarrow\Delta'',A,\Delta'}\;\mathit{Cut} \qquad \dfrac{\Gamma\Rightarrow\Delta'',B,\bot \qquad \dfrac{}{\bot,\Gamma'\Rightarrow\Delta'}\;L\bot}{\Gamma,\Gamma''\Rightarrow\Delta'',B,\Delta'}\;\mathit{Cut}}{\Gamma,\Gamma'\Rightarrow\Delta'',A\,\&\,B,\Delta'}\;R\&$$

with two cuts with lower cut-height.

- Rule$= R\vee : \Delta = \Delta'', A\vee B$. The derivation

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta'', A, B, \bot}{\Gamma \Rightarrow \Delta'', A\vee B, \bot}\,R\vee \quad \dfrac{}{\bot, \Gamma' \Rightarrow \Delta'}\,L\bot}{\Gamma, \Gamma' \Rightarrow \Delta'', A\vee B, \Delta'}\,Cut$$

is transformed into

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta'', A, B, \bot \quad \dfrac{}{\bot, \Gamma' \Rightarrow \Delta'}\,L\bot}{\Gamma, \Gamma' \Rightarrow \Delta'', A, B, \Delta'}\,Cut}{\Gamma, \Gamma' \Rightarrow \Delta'', A\vee B, \Delta'}\,R\vee$$

with a cut with lower cut-height.

- Rule$= R\supset : \Delta = \Delta'', A\supset B$. The derivation

$$\dfrac{\dfrac{A, \Gamma \Rightarrow \Delta'', B, \bot}{\Gamma \Rightarrow \Delta'', A\supset B, \bot}\,R\supset \quad \dfrac{}{\bot, \Gamma' \Rightarrow \Delta'}\,L\bot}{\Gamma, \Gamma' \Rightarrow \Delta'', A\supset B, \Delta'}\,Cut$$

is transformed into

$$\dfrac{\dfrac{A, \Gamma \Rightarrow \Delta'', B, \bot \quad \dfrac{}{\bot, \Gamma' \Rightarrow \Delta'}\,L\bot}{A, \Gamma, \Gamma' \Rightarrow \Delta'', B, \Delta'}\,Cut}{\Gamma, \Gamma' \Rightarrow \Delta'', A\supset B, \Delta'}\,R\supset$$

with a cut with lower cut-height.

(b) The cut formula is not principal in the left premiss and the left premiss is derived by L$\vee$
$\Gamma = A\vee B, \Gamma''$. The derivation

$$\dfrac{\dfrac{A, \Gamma'' \Rightarrow \Delta, D \quad B, \Gamma'' \Rightarrow \Delta, D}{A\vee B, \Gamma'' \Rightarrow \Delta, D}\,L\vee \quad D, \Gamma' \Rightarrow \Delta'}{A\vee B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta''}\,Cut$$

is transformed into

$$\dfrac{\dfrac{A, \Gamma'' \Rightarrow \Delta, D \quad D, \Gamma' \Rightarrow \Delta'}{A, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,Cut \quad \dfrac{B, \Gamma'' \Rightarrow \Delta, D \quad D, \Gamma' \Rightarrow \Delta'}{B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,Cut}{A\vee B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta''}\,L\vee$$

with two cuts with lower cut-height.

(c) The cut formula of the form $A\vee B$ or $A \& B$ is principal in both premisses.

- $D = A\vee B$, and the derivation

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A\vee B}\,R\vee \quad \dfrac{A, \Gamma' \Rightarrow \Delta' \quad B, \Gamma' \Rightarrow \Delta'}{A\vee B, \Gamma' \Rightarrow \Delta'}\,L\vee}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A, B \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, B, \Delta'}\,Cut \quad B, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta'\Delta'}\,Cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Ctr$$

with two cuts with reduced weight of the cut formula.

- $D = A \,\&\, B$, and the derivation

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \,\&\, B}\,R\& \qquad \dfrac{A, B, \Gamma' \Rightarrow \Delta'}{A \,\&\, B, \Gamma' \Rightarrow \Delta'}\,L\&}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut$$

is transformed into

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, B \qquad \dfrac{\Gamma \Rightarrow \Delta, A \quad A, B, \Gamma' \Rightarrow \Delta'}{\Gamma, B, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\,Cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Ctr$$

with two cuts with reduced weight of the cut formula.

4. Consider the quantifier rules given in the next page and explain what is wrong in the following derivations

$$\dfrac{\dfrac{[\exists x A]_2 \qquad \dfrac{[A(y/x)]_1}{\forall x A}\,\forall I}{\forall x A}\,\exists E_1}{\exists x A \supset \forall x A}\,\supset I_2 \qquad\qquad \dfrac{\dfrac{\dfrac{[\exists x A]_2 \qquad \dfrac{\dfrac{[\forall x(A \supset B)]_3}{A(y/x) \supset B}\,\forall E \quad [A(y/x)]_1}{B}\,\supset E}{B}\,\exists E_1}{\exists x A \supset B}\,\supset I_2}{\forall x(A \supset B) \supset (\exists x A \supset B)}\,\supset I_3$$

Does the additional hypothesis that B does not contain any free variables suffice to make the second one correct?

Answer: In the first inference, the introduction of the universal quantifier violates the variable restriction of the rule $\forall I$ because the conclusion depends on the open assumption $A(y/x)$ and in the general (and typical) case, y occurs free in $A(y/x)$.

In the second inference, the conclusion of the rule $\forall E$ should be $(A \supset B)(y/x)$ and also B should be $B(y/x)$ in what follows after the rule. The variable restriction of the rule $\exists E$ is then violated if formula $B(y/x)$ contains y. It is not enough to assume that B does not contain any free variables: the variable restriction also prohibits y occurring free in $\exists x A$ and in $\forall x(A \supset B)$ so we must also assume that y does not occur free in A either.

5. Consider a language without constants, nor functions. Prove that if $\exists x A$ is derivable, then $\forall x A$ is derivable.

Proof. We can assume that the derivation is in normal form. In that case, the last rule cannot be an elimination rule (because their major premisses are assumptions which would then remain open), so the last rule must be $\exists I$. If we have no constants nor functions, the substitution term in formula A must be a variable and the derivation of $\exists x A$ must be of the form

$$\dfrac{\begin{array}{c}\vdots\\ A(y/x)\end{array}}{\exists x A}\,\exists I\ .$$

In order for the derivation of $\exists x A$ to be valid, the derivation of $A(y/x)$ cannot have any open assumptions and $\exists x A$ cannot have free variables. As a consequence, the variable restriction of the rule $\forall I$ holds and we can as well derive $\dfrac{\begin{array}{c}\vdots\\ A(y/x)\end{array}}{\forall x A}\,\forall I$ □

6. Why are the following derivations incorrect?

$$\frac{\forall x \exists y(y = x+1)}{\exists y(y = y+1)}\ \forall E \qquad \frac{\sim \exists x(x \neq x)}{\exists y(\sim \exists x(x \neq y))}\ \exists I$$

Answer: The first derivation is incorrect because variable x is substituted by y which is already bound by the existential quantifier so y is not free for x in A. Thus the conclusion is not a substitution instance of formula A in the premiss as required by the standard rule $\dfrac{\forall x A}{A(t/x)}\ \forall E$. As explained on page 63 of the course book, before the substitution $[y/x]$ we should first rename the bound variable y with e.g. z and thus obtain $\exists z(z = y+1)$ in the conclusion.

The second one is incorrect for a related reason: in the rule $\dfrac{A(a/x)}{\exists x A}\ \exists I$, the premiss should be a substitution instance of A but this is not the case here. The variable x is bound by the existential quantifier and substitution does not affect bound variables. Starting from the premiss, we could only add useless quantifiers as in $\forall y(\sim\exists x(x \neq x))$, and starting from the conclusion, the substitution $[x/y]$ would have required α-conversion (renaming of bound variable x) which would have yielded $\sim\exists z(z \neq x)$ in the premiss.