

1. Prove in natural deduction for intuitionistic logic:

(a)  $\forall x \forall y A \supset \forall y \forall x A$

$$\frac{[\forall x \forall y A]^3 \quad \frac{[\forall y A(t/x)]^2 \quad [A(t/x)(u/y)]^1}{A(t/x)(u/y)} \forall E,1}{A(t/x)(u/y)} \forall E,2}{\frac{A(t/x)(u/y)}{\forall x A(u/y)} \forall I \leftarrow \text{variable restrictions hold}} \forall I \leftarrow \text{variable restrictions hold}}{\forall y \forall x A} \forall I \leftarrow \text{variable restrictions hold}}{\forall x \forall y A \supset \forall y \forall x A} \supset I,3$$

(b)  $\forall x(A \ \& \ B) \supset \forall x A \ \& \ \forall x B$

$$\frac{\frac{\forall x(A \ \& \ B) \quad \frac{[(A \ \& \ B)(t/x)]^1}{A(t/x)} \ \&E}{A(t/x)} \forall E,1}{\forall x A} \forall I}{\frac{\forall x(A \ \& \ B) \quad \frac{[(A \ \& \ B)(t/x)]^2}{B(t/x)} \ \&E}{B(t/x)} \forall E,2}{\forall x B} \forall I}{\forall x A \ \& \ \forall x B} \ \&I}{\forall x(A \ \& \ B) \supset \forall x A \ \& \ \forall x B} \supset I,3$$

(c)  $\forall x A \supset \sim \exists x \sim A$

$$\frac{[\exists x \sim A]^3 \quad \frac{[\forall x A]^4 \quad \frac{[\sim A(y/x)]^2 \quad [A(y/x)]^1}{\perp} \supset E}{\perp} \forall E,1}{\perp} \exists E,2}{\sim \exists x \sim A} \supset I,3}{\forall x A \supset \sim \exists x \sim A} \supset I,4$$

2. Assuming that  $x$  is not among the free variables of  $B$ , prove in natural deduction for intuitionistic logic:

(a)  $B \supset \forall x B$

$$\frac{\frac{[B]^1}{\forall x B} \forall I}{B \supset \forall x B} \supset I,1}{\frac{[\forall x B]^3 \quad [B]^2}{B} \forall E,2}{\forall x B \supset B} \supset I,3}{B \supset \forall x B} \ \&I$$

(b)  $B \supset \exists x B$

$$\frac{\frac{[B]^1}{\exists x B} \exists I}{B \supset \exists x B} \supset I,1}{\frac{[\exists x B]^3 \quad [B]^2}{B} \exists E,2}{\exists x B \supset B} \supset I,3}{B \supset \exists x B} \ \&I$$

(c)  $\forall x A \vee B \supset \forall x(A \vee B)$

$$\frac{\frac{\frac{\frac{[\forall x A]^2 \quad [A(t/x)]^1}{A(t/x)}{\forall E,1}}{A(t/x) \vee B} \vee I}{\forall x A \vee B} \vee I}{\frac{\frac{[B]^2}{A(t/x) \vee B} \vee I}{A(t/x) \vee B} \vee E,2}}{\frac{A(t/x) \vee B}{\forall x(A \vee B)} \vee I} \supset I,3$$

3. Find derivations of the following, both in natural deduction for classical logic and in the sequent calculus **G3c**:

(a)  $\forall x A \supset \supset \sim \exists x \sim A$

$$\frac{\frac{\frac{[\exists x \sim A]^4 \quad \perp}{\forall x A} \supset I,5}{\perp} \exists E,3}{\frac{[\forall x A]^5 \quad \frac{\frac{[\sim A(y/x)]^3 \quad [A(y/x)]^2 \quad [\perp]^1}{\perp} \supset E,2}{\perp} \supset E,1}}{[A(y/x)]} \supset I,8}{\frac{\frac{[\sim \exists x \sim A]^8 \quad \frac{[\sim A(y/x)]^7}{\exists x \sim A} \exists I}{\perp} \perp E}{A(y/x)} \text{Em,7}}{\frac{\frac{A(y/x)}{\forall x A} \vee I}{\sim \exists x \sim A \supset \forall x A} \supset I,8} \&I \quad \frac{[\perp]^1}{\perp} \supset E,6}}{\forall x A \supset \supset \sim \exists x \sim A} \&I$$

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\forall x A, A(y/x) \Rightarrow A(y/x), \perp}{\forall x A \Rightarrow A(y/x), \perp} \text{L}\forall}{\sim A(y/x), \forall x A \Rightarrow \perp} \text{L}\exists}{\exists x \sim A, \forall x A \Rightarrow \perp} \text{R}\supset}{\forall x A \Rightarrow \sim \exists x \sim A} \text{R}\supset}{\Rightarrow \forall x A \supset \sim \exists x \sim A} \text{R}\supset}}{\perp, \forall x A \Rightarrow \perp} \text{L}\perp}{\Rightarrow A(y/x), \sim A(y/x), \exists x \sim A} \text{R}\supset}{\Rightarrow A(y/x), \exists x \sim A} \text{R}\exists}{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\frac{\perp \Rightarrow A(y/x)}{\perp \Rightarrow A(y/x)} \text{L}\perp}{\sim \exists x \sim A \Rightarrow A(y/x)} \text{R}\forall}{\sim \exists x \sim A \Rightarrow \forall x A} \text{R}\supset}{\Rightarrow \sim \exists x \sim A \supset \forall x A} \text{R}\&}}{\perp \Rightarrow A(y/x)} \text{L}\perp}}{\Rightarrow \forall x A \supset \supset \sim \exists x \sim A} \text{L}\perp, \text{R}\supset, \text{R}\exists, \text{R}\forall, \text{R}\supset, \text{R}\&}$$

(b) If  $x$  is not free in  $B$ ,  $(B \supset \exists x A) \supset \exists x(B \supset A)$

$$\frac{\frac{\frac{\frac{[\exists x A]^1 \quad [B]^5}{\exists x A} \supset E,1}{\frac{[B \supset \exists x A]^6}{\exists x(B \supset A)} \supset E,2}}{\frac{[\perp]^2}{A(y/x)} \perp E}{\frac{A(y/x)}{B \supset A(y/x)} \supset I,4} \supset I,6}{\frac{\frac{[B \supset \exists x A]^6 \quad [B]^5 \quad [\exists x A]^1}{\exists x(B \supset A)} \supset E,1}{\frac{[A(y/x)]^2}{B \supset A(y/x)} \supset I}{\frac{[\sim B]^5 \quad [B]^3 \quad \frac{[\perp]^2}{A(y/x)} \perp E}{A(y/x)} \supset E,2}}{\frac{B \supset A(y/x)}{\exists x(B \supset A)} \exists I}{\frac{\frac{B \supset A(y/x)}{\exists x(B \supset A)} \exists I}{\exists x(B \supset A)} \text{Em,5}} \supset I,6$$

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{B \Rightarrow \exists x(B \supset A), B, A(t/x)}{\Rightarrow \exists x(B \supset A), B, B \supset A(t/x)} \text{R}\supset}{\Rightarrow B, \exists x(B \supset A)} \text{R}\exists}}{B, A(y/x) \Rightarrow A(y/x), \exists x(B \supset A)} \text{A}\text{x}}{\frac{A(y/x) \Rightarrow B \supset A(y/x), \exists x(B \supset A)}{A(y/x) \Rightarrow \exists x(B \supset A)} \text{R}\supset} \text{R}\exists}{\frac{A(y/x) \Rightarrow \exists x(B \supset A)}{\exists x A \Rightarrow \exists x(B \supset A)} \text{L}\exists} \text{L}\supset}{B \supset \exists x A \Rightarrow \exists x(B \supset A)} \text{L}\supset}{\Rightarrow (B \supset \exists x A) \supset \exists x(B \supset A)} \text{R}\supset$$