1. The theory of equality has the axioms

   (1) $a = a$ (reflexivity)

   (2) $a = b \,\&\, b = c \supset a = c$ (transitivity)

   (3) $a = b \supset b = a$ (symmetry)

   Show that if (2) is modified into

   (2′) $a = b \,\&\, a = c \supset b = c$

   an equivalent axiomatization is obtained.

   Proof We must show that $(1) \,\&\, (2) \,\&\, (3)$ can be derived from $(1) \,\&\, (2') \,\&\, (3)$ and vice versa. Actually (3) follows already from (1) and (2') as we will see below. We will proceed informally:

   "⇒" Let us assume $a = b \,\&\, a = c$ and try to get $b = c$ without using axiom (2'). From (3) we get $b = a \,\&\, a = c$. From (2) we get $b = c$.

   "⇐" Assume $a = b$ and try to get $b = a$ without (2) and (3): From (1) we get $a = b \,\&\, a = a$ and from (2') we get $b = a$. So we have symmetry.

   Let us then assume $a = b \,\&\, b = c$ and try to get $a = c$ without (2). We get $b = a$ from (1) and (2') as above so we have $b = a \,\&\, b = c$. With (2') we get $a = c$.

2. Give the nonlogical rules corresponding to the axioms for equality in such a way that, by extending **G3c** with these rules, a cut- and contraction-free sequent calculus **G3Eq** is obtained. Derive symmetry from the rules corresponding to (1) and (2').

   Following the general rule scheme given in Chapter 6 of the book, we get the following:

   (i) From $a = a$ we get the rule

   $$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{Refl}$$

   (ii) From $a = b \,\&\, b = c \supset a = c$ we get

   $$\frac{a = b, b = c, a = c, \Gamma \Rightarrow \Delta}{a = b, b = c, \Gamma \Rightarrow \Delta}\ \textit{Trans}$$

   Here we must check the closure condition 6.1.7. Substitution [a/b,a/c] leads to the following instance where the same active formula appears twice in the conclusion:

   $$\frac{a = a, a = a, a = a, \Gamma \Rightarrow \Delta}{a = a, a = a, \Gamma \Rightarrow \Delta}\ \textit{Trans}$$

   We should thus add to the system the rule

   $$\frac{a = a, a = a, \Gamma, \Delta}{a = a, \Gamma \Rightarrow \Delta}$$

   but this is an instance of the rule *Refl* so we don't need it.

   (iii) From $a = b \supset b = a$ we get the rule

   $$\frac{a = b, b = a, \Gamma \Rightarrow \Delta}{a = b, \Gamma \Rightarrow \Delta}\ \textit{Sym}$$

(iv) If instead of axioms (2) and (3) we had used (2') $a = b \,\&\, a = c \supset b = c$ we would have got

$$\frac{a = b, a = c, b = c, \Gamma \Rightarrow \Delta}{a = b, a = c, \Gamma \Rightarrow \Delta}\ \textit{Eucl}$$

Here if b and c are identical, we will get a duplicate in the conclusion but the rule suggested by the closure condition is again an instance of the rule *Refl*:

$$\frac{a = b, b = b, \Gamma \Rightarrow \Delta}{a = b, \Gamma \Rightarrow \Delta}$$

The nonlogical rules follow the general rule-scheme and satisfy the closure condition so according to Theorems 6.2.2 and 6.2.3, contraction and cut are admissible.

Symmetry can be derived in the following way:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{a = b, a = a, b = a \Rightarrow b = a}\ \textit{Ax}}{a = b, a = a \Rightarrow b = a}\ \textit{Eucl}}{a = b \Rightarrow b = a}\ \textit{Refl}}{\Rightarrow a = b \supset b = a}\ R\supset$$

3. Using proof search in **G3Eq**, show that the axioms (1), (2), (3) (or (1), (2′)) are independent of each other, meaning that none of them follows from the remaining ones.

Proof For each axiom A, we must show that an attempt to derive $\Rightarrow A$ without the rule corresponding to A will fail:

- $\Rightarrow a = a$: By looking at the rules of **G3c** (page 49 in the book) we see that there is no rule that can be used to derive a sequent where the left-hand side of the sequent arrow is empty and on the right-hand side there is an atomic formula. Also the rules *Sym* and *Trans* require that the left-hand side of the arrow is not empty so $\Rightarrow a = a$ can not be derived without using *Refl*.
- $\Rightarrow a = b \,\&\, b = c \supset a = c$: We can apply the rules $R\supset$ and $L\&$ to this sequent to get the following (the order of the application does not matter):

$$\cfrac{\cfrac{a = b, b = c \Rightarrow a = c}{a = b \,\&\, b = c \Rightarrow a = c}\ L\&}{\Rightarrow a = b \,\&\, b = c \supset a = c}\ R\supset$$

  At any point in the derivation, we can also apply rules *Refl* and *Sym* to add occurrences of formulas $b = a, c = b$ or formulas of the form $a = a, b = b, c = c, \ldots$ to the left-hand side of the sequent arrow. However, we cannot obtain axioms, since that would require getting $a = c$ to the left-hand side.
- $\Rightarrow a = b \supset b = a$: The only logical rule we can apply is $R\supset$:

$$\frac{a = b \Rightarrow b = a}{\Rightarrow a = b \supset b = a}\ R\supset$$

  Rule *Refl* can be used to add formulas $a = a$ and $b = b$ and *Trans* can be used to add instances of $a = a, b = b$ and $a = b$ but not $b = a$ which would be needed in order to apply *Ax*. □

4. Prove that in the extension **G3*** of **G3c** with rules following the general rule-scheme, all the nonlogical rules permute down with respect to the logical rules.

   Proof. We will have to show that any derivation where a nonlogical rule is preceded by some logical rule of **G3c**, can be transformed into a form where the nonlogical rule comes before the logical rule. For clarity, we will first look at the simple case where the nonlogical rule only has one premiss. Consider first the case with the logical rule $L\,\&$: The derivation

$$\dfrac{\dfrac{\vec{P},Q,A,B,\Gamma \Rightarrow \Delta}{\vec{P},Q,A\,\&\,B,\Gamma \Rightarrow \Delta}\,L\&}{\vec{P},A\,\&\,B,\Gamma \Rightarrow \Delta}\,\textit{Rule-scheme}$$

   can be transformed to

$$\dfrac{\dfrac{\vec{P},Q,A,B,\Gamma \Rightarrow \Delta}{\vec{P},A,B,\Gamma \Rightarrow \Delta}\,\textit{Rule-scheme}}{\vec{P},A\,\&\,B,\Gamma \Rightarrow \Delta}\,L\&$$

   Let us consider whether the variable restrictions pose any problems by taking the logical rule to be $R\forall$:

$$\dfrac{\dfrac{\vec{P},Q,\Gamma \Rightarrow \Delta,A(y/x)}{\vec{P},Q,\Gamma \Rightarrow \Delta,\forall xA}\,R\forall}{\vec{P},\Gamma \Rightarrow \Delta,\forall xA}\,\textit{Rule-scheme}$$

   can be transformed into

$$\dfrac{\dfrac{\vec{P},Q,\Gamma \Rightarrow \Delta,A(y/x)}{\vec{P},\Gamma \Rightarrow \Delta,A(y/x)}\,\textit{Rule-scheme}}{\vec{P},\Gamma \Rightarrow \Delta,\forall xA}\,R\forall$$

   The variable restriction poses no problems because from the validity of the first derivation it follows that $x \notin FV(\vec{P},Q,\Gamma,\Delta)$ and thus it is the case that $x \notin FV(\vec{P},\Gamma,\Delta)$ which is required for the second derivation.

   The other cases with just one premiss are similar. Let us then consider the general case with n premisses. Let the k:th premiss be derived using a logical rule:

$$\dfrac{Q_1,\vec{P},\Gamma \Rightarrow \Delta \quad \dots \quad \dfrac{Q_k,\vec{P},\Gamma' \Rightarrow \Delta'}{Q_k,\vec{P},\Gamma \Rightarrow \Delta}\,\textit{logical} \quad \dots \quad Q_n,\vec{P},\Gamma \Rightarrow \Delta}{\vec{P},\Gamma \Rightarrow \Delta}\,\textit{Rule-scheme}$$

   If the logical rule is $R\vee$, for instance, we have the derivation

$$\dfrac{Q_1,\vec{P},\Gamma \Rightarrow \Delta,A\vee B \quad \dots \quad \dfrac{Q_k,\vec{P},\Gamma \Rightarrow \Delta,A,B}{Q_k,\vec{P},\Gamma \Rightarrow \Delta,A\vee B}\,R\& \quad \dots \quad Q_n,\vec{P},\Gamma \Rightarrow \Delta,A\vee B}{\vec{P},\Gamma \Rightarrow \Delta,A\vee B}\,\textit{Rule-scheme}$$

   By invertibility of the rule $R\vee$, because the premisses 1 to $k-1$ and $k+1$ to n are derivable and of the form $Q_i,\vec{P},\Gamma \Rightarrow \Delta,A\vee B$ then also the corresponding premisses of the form $Q_i,\vec{P},\Gamma \Rightarrow \Delta,A,B$ are derivable. Thus, we can transform the derivation into

$$\dfrac{\dfrac{Q_1,\vec{P},\Gamma \Rightarrow \Delta,A,B \quad \dots \quad Q_k,\vec{P},\Gamma \Rightarrow \Delta,A,B \quad \dots \quad Q_n,\vec{P},\Gamma \Rightarrow \Delta,A,B}{\vec{P},\Gamma \Rightarrow \Delta,A,B}\,\textit{Rule-scheme}}{\vec{P},\Gamma \Rightarrow \Delta,A\vee B}\,R\&$$

If the logical rule is $L\supset$, we have the derivation

$$\dfrac{Q_1,\vec{P},A\supset B,\Gamma\Rightarrow\Delta \quad \ldots \quad \dfrac{Q_k,\vec{P},\Gamma\Rightarrow\Delta,A \quad B,Q_k,\vec{P},\Gamma\Rightarrow\Delta}{Q_k,\vec{P},A\supset B,\Gamma\Rightarrow\Delta}\,L\supset \ldots \quad Q_n,\vec{P},A\supset B,\Gamma\Rightarrow\Delta}{\vec{P},A\supset B,\Gamma\Rightarrow\Delta}\,\textit{Rule-scheme}$$

By invertibility of the rule $L\supset$, sequents of the form $Q_i,\vec{P},\Gamma\Rightarrow\Delta,A$ and $B,Q_i,\vec{P},\Gamma\Rightarrow\Delta$ are derivable for all $i\in\{1,\ldots,k-1,k+1,\ldots n\}$ so we get the derivation

$$\dfrac{\dfrac{Q_1,\vec{P},\Gamma\Rightarrow\Delta,A \quad \ldots \quad Q_k,\vec{P},\Gamma\Rightarrow\Delta,A \quad \ldots \quad Q_n,\vec{P},\Gamma\Rightarrow\Delta,A}{\vec{P},\Gamma\Rightarrow\Delta,A}\,\textit{Rule-scheme} \quad \dfrac{B,Q_1,\vec{P},\Gamma\Rightarrow\Delta \quad \ldots \quad B,Q_k,\vec{P},\Gamma\Rightarrow\Delta \quad \ldots \quad B,Q_n,\vec{P},\Gamma\Rightarrow\Delta}{B,\vec{P},\Gamma\Rightarrow\Delta}\,\textit{Rule-scheme}}{\vec{P},A\supset B,\Gamma\Rightarrow\Delta}\,L\supset$$

The other cases are similar.