1. Show that in **G3cp** extended with the axioms for the theory of partial order (page 146 in the book) minimal derivations satisfy the subterm property (in other words, all terms in the minimum-size derivation of $\Gamma \Rightarrow \Delta$ are subterms of terms in Γ, Δ).

   Proof Since **G3cp** satisfies the subformula property (Corollary 3.2.4, page 57), it is enough to consider the new rules:

$$\frac{a \le a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ref} \qquad \frac{a \le c, a \le b, b \le c, \Gamma \Rightarrow \Delta}{a \le b, b \le c, \Gamma \Rightarrow \Delta}\ \mathit{Trans}$$

   In *Trans*, all the terms in the premiss are subterms of the terms in the conclusion so no term can disappear in an application of *Trans*. On the other hand, *Ref* seems problematic because $a \le a$ disappears. Let us consider whether a term a can disappear in a mimimum-size derivation by considering the possible derivations where *Ref* is used.

   Either the formula $a \le a$ has been active in the derivation before disappearing or it has not been. If it has not been active, it plays no role in the derivation and can be removed altogether so the original derivation is not minimal. If it has been active, there are two possibilities where it can have been active: either in *Ax* or in *Trans* (because in all other rules, active formulas are complex).

   (1) If *Ax*, then the axiom has been of the form $a \le a, \Gamma' \Rightarrow \Delta', a \le a$ and a appears in the succedent. It cannot disappear from the derivation because i) it will either stay on the right hand side of the sequent arrow or ii) it will be transferred to the left hand side in an application of $L \supset$ in a complex form $a \le a \supset B$ and cannot disappear from there (because only atomic formulas can disapper in *Ref*).

   (2) If *Trans*, then it can only appear in one of the three following forms:

$$\frac{a \le b, a \le a, a \le b, \Gamma' \Rightarrow \Delta'}{a \le a, a \le b, \Gamma' \Rightarrow \Delta'} \quad \frac{b \le a, b \le a, a \le a, \Gamma' \Rightarrow \Delta'}{b \le a, a \le a, \Gamma' \Rightarrow \Delta'} \quad \frac{a \le a, a \le a, a \le a, \Gamma' \Rightarrow \Delta'}{a \le a, a \le a, \Gamma' \Rightarrow \Delta'}$$

   In all these cases, there is a duplicated atom that is removed and by height-preserving contraction, the conclusion is derivable in the same height without the rule so the derivation cannot be mimimal contrary to the assumption.

   Therefore, *Ref* cannot make a term disappear in a mimum-height derivation. $\square$

2. Show that $\Box A \supset \Box\Box A$ is valid in a Kripke model $\langle W, R, v \rangle$ for an arbitrary evaluation v iff R is transitive. Give an example of a model where $\Box A \supset \Box\Box A$ is not true.

   Proof

   "$\Rightarrow$" Let us consider a world $w \in W$ and any valuation v such that $x \Vdash A$ iff wRx. Thus $w \Vdash \Box A$ and $w \Vdash \Box\Box A$ because $\Box A \supset \Box\Box A$ is valid. So it must be the case that for all x, y such that wRx and xRy, $y \Vdash A$. According to the choice of valuation, also wRy and R must be transitive.

   "$\Leftarrow$" Suppose $x \Vdash \Box A$. For any y such that xRy, $y \Vdash A$. Because R is transitive, also for any z s.t. yRz, $z \Vdash A$. Then $y \Vdash \Box A$ and thus $x \Vdash \Box\Box A$. $\square$

   The axiom $\Box A \supset \Box\Box A$ is not valid in the model $\langle \{a,b,c\}, \{(a,b),(b,c)\}, \{(A,\{b\})\} \rangle$ since $x \Vdash \Box A$ but $x \nVdash \Box\Box A$.

3. Show that the following formulas are valid in every model:

   (a) $\Box A \,\&\, \Box B \supset \Box(A \,\&\, B)$

   Let $\langle W, R, v\rangle$ be a Kripke model and suppose that $x \in W$ and $x \Vdash \Box A \,\&\, \Box B$. This means that in every world y such that xRy, $y \Vdash A$ and $y \Vdash B$. Hence, $y \Vdash A \,\&\, B$ and $x \Vdash \Box(A \,\&\, B)$.

   (b) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

   Let $\langle W, R, v\rangle$ be a Kripke model and suppose that $x \in W$ and $x \Vdash \Box(A \supset B)$. Then in every world y such that xRy, $y \Vdash A \supset B$. Thus, if $x \Vdash \Box A$ then $y \Vdash A$ which together with $y \Vdash A \supset B$ implies $y \Vdash B$. Thus $x \Vdash \Box B$ and, consequently, $x \Vdash \Box(A \supset B) \supset (\Box A \supset \Box B)$.

4. Show that if R is symmetric and transitive then $\Diamond A \supset \Box\Diamond A$ is valid.

   Proof Let R be symmetric and transitive. Consider world x s.t. $x \Vdash \Box A$. This means that there exists a world y s.t. xRy and $y \Vdash A$. Consider then an arbitrary world z s.t. xRz. Because R is symmetric and transitive, it follows from yRx and xRz that yRz and zRy so $x \Vdash \Diamond A$. Thus $x \Vdash \Box\Diamond A$. $\Box$

5. (Extra exercise) Show that $\Diamond\Box A \supset \Box\Diamond A$ is valid in a Kripke model $\langle W, R, v\rangle$ for an arbitrary v iff R is directed. (Directedness means that if xRy and xRz, then there exists u such that yRu and zRu.)

   Proof

   "$\Rightarrow$" Suppose xRy, xRz and let $w \Vdash A \Leftrightarrow yRw$. Then $y \Vdash \Box A$ and $x \Vdash \Diamond\Box A$. By hypothesis then $x \Vdash \Box\Diamond A$ and therefore $z \Vdash \Diamond A$. Thus zRw for some $w \Vdash A$ for which yRw so z and y have a common successor.

   "$\Leftarrow$" If $x \Vdash \Diamond\Box A$ then there exists a world y such that xRy and $y \Vdash \Box A$ i.e. for all u s.t. yRu, $u \Vdash A$. Consider an arbitrary world z s.t. xRz. Because R is directed, z and y have a common successor and $z \Vdash \Diamond A$. Thus $x \Vdash \Box\Diamond A$. $\Box$