On Reduced Beltrami Equations and Linear Families of Quasiregular Mappings

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Quasiregular Mappings

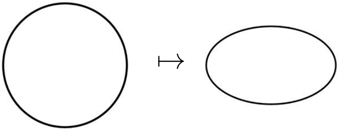
As a reminder, $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$, $\Omega \subset \mathbb{C}$ a domain, is K-*quasiregular* if the classical Beltrami equation holds for almost every $z \in \Omega$

$$\partial_{\bar z} f(z) = \mu(z)\partial_z f(z), \qquad |\mu(z)| \leqslant k < 1, \quad K = \frac{1+k}{1-k},$$

where $\partial_{\bar z} f(z) = \frac{1}{2}\big(\partial_x f(z) + i\partial_y f(z)\big)$ and $\partial_z f(z) = \frac{1}{2}\big(\partial_x f(z) - i\partial_y f(z)\big)$.

If, in addition, the mapping is also a homeomorphism, then it is called *quasiconformal*.

Infinitesimally a quasiconformal function maps circles into ellipses.



Reduced Beltrami Equation

Mapping $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$, solves *reduced Beltrami equation*, if

$$\partial_{\bar z} f(z) = \lambda(z)\,\mathrm{Im}\big(\partial_z f(z)\big), \qquad |\lambda(z)| \leqslant k < 1,$$

for almost every $z \in \Omega$.

- A differential constraint is stronger than the one in the Beltrami equation, hence f is K-quasiregular with $K = \frac{1+k}{1-k}$.
- $\mathcal{J}(z, f) := \mathrm{Im}\big(\partial_z f\big)$ is a null Lagrangian.

Generalized Stoïlow Factorization

Theorem (Astala, Iwaniec, and Martin, 2009)

Let $\Phi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a homeomorphic solution to the general Beltrami equation

$$\partial_{\bar z} g(z) = \mu(z)\partial_z g(z) + \nu(z)\overline{\partial_z g(z)}, \qquad |\mu(z)| + |\nu(z)| \leqslant k < 1,$$

for almost every $z \in \Omega$. Then any other solution $\Psi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ takes the form

$$\Psi = F \circ \Phi,$$

where F solves the reduced Beltrami equation in $\Phi(\Omega)$ with

$$\lambda(w) = \frac{-2i\,\nu(z)}{1 + |\nu(z)|^2 - |\mu(z)|^2}, \qquad w = \Phi(z), \quad z \in \Omega.$$

Also the converse direction holds.

On Reduced Beltrami Equations

The following answers in positive a conjecture of Astala, Iwaniec, and Martin.

Theorem

Suppose $f : \Omega \to \mathbb{C}$, $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$, *is a solution to the reduced Beltrami equation*

$$\partial_{\bar{z}} f(z) = \lambda(z)\,\mathrm{Im}\big(\partial_z f(z)\big), \qquad |\lambda(z)| \leqslant k < 1, \qquad a.e.\ z \in \Omega.$$

Then either $\partial_z f$ *is a constant or else*

$$\mathrm{Im}\big(\partial_z f\big) \neq 0 \qquad \textit{almost everywhere in } \Omega.$$

Thus if $\mathrm{Im}(\partial_z f)$ *vanishes on a set of positive measure, then* $f(z) = az + b$, *where* $a \in \mathbb{R}$ *and* $b \in \mathbb{C}$.

What Was Known?

Theorem

Suppose $f : \Omega \to \mathbb{C}$, $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$, *is a solution to the reduced Beltrami equation*

$$\partial_{\bar{z}} f(z) = \lambda(z)\,\mathrm{Im}\big(\partial_z f(z)\big), \qquad |\lambda(z)| \leqslant k < 1, \qquad a.e.\ z \in \Omega.$$

Then either $\partial_z f$ *is a constant or else* $\mathrm{Im}\big(\partial_z f\big) \neq 0$ *almost everywhere in* Ω.

- (Giannetti, Iwaniec, Kovalev, Moscariello, and Sbordone, 2004) proved for homeomorphisms of the plane $\mathbb{C}$, when $k < 1/2$.
- (Alessandrini and Nesi, 2009) for homeomorphisms of the plane $\mathbb{C}$

Linear Families of Quasiregular Mappings

Given an $\mathbb{R}$-linear subspace $\mathcal{F} \subset W^{1,2}_{\mathrm{loc}}(\Omega)$, $\mathcal{F}$ is a *linear family of quasiregular mappings*, if there is $1 \leqslant K < \infty$ such that for every $g \in \mathcal{F}$ the function g is K-quasiregular in Ω.

The family $\mathcal{F}$ is *generated* by the maps Φ and Ψ if

$$\mathcal{F} = \{a\,\Phi + b\,\Psi : \; a, b \in \mathbb{R}\}$$

for some quasiregular mappings $\Phi : \Omega \to \mathbb{C}$ and $\Psi : \Omega \to \mathbb{C}$.

In case of linear families that consist of quasiconformal mappings, $\dim \mathcal{F} \leqslant 2$, (Bojarski, D'Onofrio, Iwaniec, and Sbordone, 2005).

Recall that a linear family of quasiregular mappings is not always two-dimensional; for instance, 1-quasiregular family spanned by functions $f_1(z) = z$, $f_2(z) = z^2$, and $f_3(z) = z^3$.

Linear Families of Quasiregular Mappings

In general, quasiregularity is not preserved under linear combinations; simple example is $f(z) = k\bar{z} + z$, $g(z) = k\bar{z} - z$.

However, if we have mappings that happen to be solutions to the same *general Beltrami equation*

$$\partial_{\bar{z}} g(z) = \mu(z)\partial_z g(z) + \nu(z)\overline{\partial_z g(z)}, \qquad |\mu(z)| + |\nu(z)| \leqslant k < 1,$$

for almost every $z \in \Omega$, then their linear combinations are quasiregular.

Theorem

For any linear two-dimensional family $\mathcal{F}$ of quasiregular mappings in a domain $\Omega \subset \mathbb{C}$ there exists a corresponding general Beltrami equation satisfied by every element $g \in \mathcal{F}$.

Moreover, the associated equation is unique.

What Was Known?

Theorem

For any linear two-dimensional family $\mathcal{F}$ of quasiregular mappings in a domain $\Omega \subset \mathbb{C}$ there exists a corresponding general Beltrami equation satisfied by every element $g \in \mathcal{F}$. The associated equation is unique.

- Existence, (Bojarski, D'Onofrio, Iwaniec, and Sbordone, 2005); uniquely defined on the regular set
- Uniqueness for family of K-quasiconformal mappings, $1 \leqslant K < 3$ (Giannetti, Iwaniec, Kovalev, Moscariello, and Sbordone, 2004); $1 \leqslant K < \infty$ (Alessandrini and Nesi, 2009); the singular set has measure zero for homeomorphisms
- The homeomorphic case with the ideas developed in (Bojarski, D'Onofrio, Iwaniec, and Sbordone, 2005) and (Giannetti, Iwaniec, Kovalev, Moscariello, and Sbordone, 2004): the family of Beltrami differential operators, $1 \leq K < \infty$, in a domain is G-compact.

Idea of the Proof: Existence

For any linear two-dimensional family $\mathcal{F}$ of quasiregular mappings in a domain $\Omega \subset \mathbb{C}$ there exists a corresponding general Beltrami equation satisfied by every element $g \in \mathcal{F}$.

Assume $\Phi, \Psi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ are generators. The goal is to find coefficients μ and ν such that

$$\partial_{\bar{z}}\Phi = \mu\partial_z\Phi + \nu\overline{\partial_z\Phi} \qquad \text{and} \qquad \partial_{\bar{z}}\Psi = \mu\partial_z\Psi + \nu\overline{\partial_z\Psi}, \tag{1}$$

almost everywhere in Ω.

In the *regular set* $\mathcal{R}_{\mathcal{F}}$ of $\mathcal{F}$, i.e., the set of points $z \in \Omega$ where the matrix

$$M(z) = \begin{bmatrix} \partial_z\Phi(z) & \overline{\partial_z\Phi(z)} \\ \partial_z\Psi(z) & \overline{\partial_z\Psi(z)} \end{bmatrix}$$

is invertible, the values $\mu(z)$ and $\nu(z)$ are uniquely determined by (1):

Idea of the Proof: Existence (cont.)

$$\mu = i\,\frac{\Psi_{\bar z}\overline{\Phi_z} - \overline{\Psi_z}\Phi_{\bar z}}{2\,\mathrm{Im}\big(\Phi_z\overline{\Psi_z}\big)}, \qquad \nu = i\,\frac{\Phi_{\bar z}\Psi_z - \Phi_z\Psi_{\bar z}}{2\,\mathrm{Im}\big(\Phi_z\overline{\Psi_z}\big)}.$$

Note that changing the generators corresponds to multiplying

$$M(z) = \begin{bmatrix} \partial_z\Phi(z) & \overline{\partial_z\Phi(z)} \\ \partial_z\Psi(z) & \overline{\partial_z\Psi(z)} \end{bmatrix}$$

by an invertible constant matrix.

Hence the regular set and its complement, the *singular set*

$$\mathcal{S}_{\mathcal{F}} = \big\{z \in \Omega : 2i\,\mathrm{Im}\big(\Phi_z(z)\overline{\Psi_z(z)}\big) = \det M(z) = 0\big\}, \qquad \blacksquare$$

depend only on the family $\mathcal{F}$ and not the choice of generators.

Idea of the Proof: Existence (cont.)

On the singular set it can be proven that for almost every $z \in \mathcal{S}_{\mathcal{F}}$ the vector $\big(\Phi_{\bar z}(z), \Psi_{\bar z}(z)\big)$ lies in the range of the linear operator $M(z) : \mathbb{C}^2 \to \mathbb{C}^2$.

It follows that on the singular set one may define $\nu(z) = 0$.

Here the assumption that the family $\mathcal{F}$ consists entirely of quasiregular mappings is needed. By quasiregularity, one has for every $\alpha, \beta \in \mathbb{R}$

$$|\alpha\, \partial_{\bar z}\Phi(z) + \beta\, \partial_{\bar z}\Psi(z)| \leqslant k|\alpha\, \partial_z\Phi(z) + \beta\, \partial_z\Psi(z)|, \qquad \text{for a.e. } z \in \Omega. \quad (2)$$

Finally, ellipticity bounds follow for the singular set $\mathcal{S}_{\mathcal{F}}$ by definition of μ and ν, since Φ and Ψ are K-quasiregular.

For the regular set one tests the inequality (2) by real-valued measurable functions $\theta(z)$ instead of parameters α and β.

Uniqueness and Reduced Beltrami Equations

To show the uniqueness, we prove that the singular set

$$\mathcal{S}_{\mathcal{F}} = \left\{ z \in \Omega : 2i\,\mathrm{Im}\left(\Phi_z(z)\overline{\Psi_z(z)}\,\right) = \det M(z) = 0 \right\}$$

has measure zero.

Here reduced Beltrami equations come into play.

We can assume Φ is nonconstant. As a nonconstant quasiregular mapping, Φ has the branch set that consists of isolated points; it is enough to study points outside the branch set.

Let z_0 be such a point. There exists a ball $B := \mathbb{D}(z_0, r)$ such that $\Phi|_B : B \to \Phi(B)$ is a homeomorphism, hence quasiconformal. From the generalized Stoïlow factorization we know that $\Psi = F \circ \Phi$ in B, where F solves the reduced Beltrami equation in $\Phi(B)$.

Uniqueness and Reduced Beltrami Equations (cont.)

Let $z \in B$. Using the chain rule and a straightforward calculation gives

$$J(z,\Phi)\,\mathrm{Im}(F_w \circ \Phi) = (-1 + |\mu|^2 - |\nu|^2)\,\mathrm{Im}(\Phi_z \overline{\Psi_z}).$$

Since Φ preserves sets of zero measure, the statement,

$$\mathcal{S}_{\mathcal{F}} = \left\{ z \in \Omega : 2i\,\mathrm{Im}\left(\Phi_z(z)\overline{\Psi_z(z)}\,\right) = \det M(z) = 0 \right\}$$

has measure zero, follows by by the theorem for reduced Beltrami equations.

Idea of the Proof

Theorem

Suppose $f : \Omega \to \mathbb{C}$, $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$, *is a solution to the reduced Beltrami equation*

$$\partial_{\bar{z}} f(z) = \lambda(z)\,\mathrm{Im}\big(\partial_z f(z)\big), \qquad |\lambda(z)| \leqslant k < 1, \qquad a.e.\ z \in \Omega.$$

Then either $\partial_z f$ *is a constant or else* $\mathrm{Im}\big(\partial_z f\big) \neq 0$ *almost everywhere in* Ω.

Assume $|E| := |\{z \in \Omega : \mathrm{Im}\big(\partial_z f(z)\big) = 0\}| > 0$.

Goal: For a.e. $z_0 \in E$, $f(w) = c_0 + c_1\,(w - z_0) + \mathcal{E}(w)$ near the point z_0, where $c_0 \in \mathbb{C}$, $c_1 \in \mathbb{R}$ are constants depending only on f and z_0, and

$$\int_{\mathbb{D}(z_0,r)} |D\mathcal{E}|\,dm = \mathcal{O}(r^{n+1})$$

holds for small enough $r > 0$ and for all positive integers n.

From the Goal to the Statement (Idea of the Proof (cont.))

The constant c_1 is real and hence $g(w) := f(w) - c_0 - c_1(w - z_0)$ solves the same reduced Beltrami equation as f. Therefore, g is quasiregular with

$$\left(\int_{\mathbb{D}(z_0,r)} |Dg|^2 dm \right)^{1/2} = \mathcal{O}(r^{N+1}), \qquad \text{when } r \text{ is small enough,}$$

for all positive integers N. We have the *Hölder continuity* of the form

$$|g(z_0) - g(w)| \leqslant c \left(\frac{|z_0 - w|}{r} \right)^{\alpha(K)} \left(\int_{\mathbb{D}(z_0,r)} |Dg|^2 dm \right)^{1/2},$$

$w \in \mathbb{D}(z_0, r/2)$ and $0 < \alpha(K) < 1$. Thus

$$\sup_{|z_0 - w| < r/2} |g(z_0) - g(w)| = \mathcal{O}(r^{N+1}).$$

This proves our statement: If g is nonconstant, there is a contradiction, since the classical Stoïlow factorization gives

$$cr^{\gamma} \leqslant \sup_{|z_0 - w| < r/2} |g(z_0) - g(w)|, \qquad \gamma > 0.$$

Stages to the Goal (Idea of the Proof (cont.))

Goal: For a.e. $z_0 \in E$, $f(w) = c_0 + c_1(w - z_0) + \mathcal{E}(w)$ near the point z_0, where $c_0 \in \mathbb{C}$, $c_1 \in \mathbb{R}$ are constants depending only on f and z_0 and

$$\int_{\mathbb{D}(z_0,r)} |D\mathcal{E}| dm = \mathcal{O}(r^{n+1})$$

holds for small enough $r > 0$ and for all positive integers n.

- an adjoint equation approach and a weak reverse Hölder inequality for the convergence rate of the integral of the derivative (almost every $z_0 \in E$ is a zero of infinite order)
- a series representation by generalized Cauchy formula

Same Zeros (Idea of the Proof (cont.))

Mapping $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a solution to the reduced Beltrami equation

$$\partial_{\bar{z}} f(z) = \lambda(z)\,\mathrm{Im}\big(\partial_z f(z)\big), \qquad |\lambda(z)| \leqslant k < 1,$$

for almost every $z \in \Omega$. Let us write $f(z) = u(z) + iv(z)$, where u and v are real-valued.

Taking the imaginary part of the reduced equation gives

$$2\,\mathrm{Im}\big(\partial_z f(z)\big) = v_x - u_y = \frac{2}{\mathrm{Im}(\lambda)+1} v_x = \frac{2}{\mathrm{Im}(\lambda)-1} u_y.$$

Since $|\,\mathrm{Im}(\lambda(z))| \leqslant |\lambda(z)| \leqslant k < 1$, the coefficients $2/(\mathrm{Im}(\lambda(z)) \pm 1)$ are uniformly bounded from below. Hence $\mathrm{Im}(\partial_z f)$ and u_y have the same zeros.

Adjoint Equation (Idea of the Proof (cont.))

Mapping u_y is a real-valued weak solution to the *adjoint equation* $L^*(u_y) = 0$; this means

$$\int_\Omega u_y L(\varphi) dm = 0, \qquad \text{for every } \varphi \in C_0^\infty(\Omega).$$

We set as a non-divergence type, uniformly elliptic operator L

$$L = \frac{\partial^2}{\partial x^2} + a_{12}\frac{\partial^2}{\partial x \partial y} + a_{22}\frac{\partial^2}{\partial y^2}, \qquad a_{12} = \frac{2\,\mathrm{Re}(\lambda)}{1 - \mathrm{Im}(\lambda)}, \quad a_{22} = \frac{1 + \mathrm{Im}(\lambda)}{1 - \mathrm{Im}(\lambda)}.$$

Key point: recall that the components of solutions $f = u + iv$ to general Beltrami equations satisfy a divergence type second-order equation; now,

$$\mathrm{div}\, A\nabla u = 0, \qquad A(z) := \begin{bmatrix} 1 & a_{12}(z) \\ 0 & a_{22}(z) \end{bmatrix}.$$

Weak Reverse Hölder Inequality (Idea of the Proof (cont.))

Theorem

Let $\omega \in L^2_{\mathrm{loc}}(\Omega)$ be a real-valued weak solution to the adjoint equation $L^*(\omega) = 0$. Then a weak reverse Hölder inequality holds for ω; namely,

$$\left(\frac{1}{r^2} \int_B \omega^2 dm \right)^{1/2} \leqslant \frac{c}{r^2} \int_{2B} |\omega| dm,$$

for every disk $B := \mathbb{D}(a, r)$ such that $2B := \mathbb{D}(a, 2r) \subset \Omega$. The constant c depends only on the ellipticity constant K.

There is a stronger result for non-negative solutions: a reverse Hölder inequality holds (Fabes and Stroock, 1984); this was used in the case of homeomorphisms of the plane. ∎

Weak Reverse Hölder Inequality (Idea of the Proof (cont.))

$$\left(\frac{1}{r^2}\int_B \omega^2 dm\right)^{1/2} \leqslant \frac{c}{r^2}\int_{2B} |\omega| dm$$

Key points:

- We solve the *Dirichlet problem*

$$L(g) = h, \qquad h \in L^2(\mathcal{D}), \qquad g \in W^{2,2}(\mathcal{D}) \quad \text{with } g = 0 \text{ on } \partial\mathcal{D},$$

for $\mathcal{D} = 2\mathbb{D}$ and $h = \omega\chi_{\mathbb{D}} \in L^2(2\mathbb{D})$.

- Let $1 < \delta < 4/3$ and $\varphi \in C_0^\infty\big((3/2)\delta\mathbb{D}\big)$ satisfy $\varphi \equiv 1$ on $\delta\mathbb{D}$.

$$\int_{\mathbb{D}} \omega^2 = \int_{2\mathbb{D}} \omega L(g)\varphi = -2\int_{2\mathbb{D}} \omega\langle A\nabla\varphi, \nabla g\rangle - \int_{2\mathbb{D}} \omega g L(\varphi) \qquad \blacksquare$$

- If $L(g) = 0$ in a subdomain $V \subset \mathcal{D}$, then the complex gradient g_z is quasiregular in V; plus, norm estimates for every relatively compact smooth subdomain $V' \subset V$ (Astala, Iwaniec, and Martin, 2006).

Zeros of Infinite Order (Idea of the Proof (cont.))

Theorem (Bojarski and Iwaniec, 1983)

Let ω satisfy a weak reverse Hölder inequality. Then, for almost every zero z_0 of ω and for every positive integer N, there is $r_0(z_0, N) > 0$ such that

$$\int_{\mathbb{D}(z_0,r)} |\omega| dm \leqslant \frac{r^N}{r_0^N} \int_{\mathbb{D}(z_0,2r_0)} |\omega| dm = \mathcal{O}(r^N), \qquad 0 < r \leqslant r_0(z_0, N).$$

- Let z_0 be a point of density of $E = \{z \in \Omega : \omega(z) = 0\}$. Since z_0 is a density point, for $r_0 := r_0(z_0, N)$ sufficiently small, $0 < \delta \leqslant 1$,

$$|\mathbb{D}(z_0, \delta r_0) \setminus E| \leqslant \frac{(\delta r_0)^2}{c^2\, 2^{2N}},$$

  where c is the constant from the weak reverse Hölder inequality.
- Using the weak reverse Hölder inequality and iterating gives our claim.

Series Representation (Idea of the Proof (cont.))

The adjoint equation approach with zeros of infinite order gives

$$\int_{\mathbb{D}(z_0,r)} |\partial_{\bar{z}} f| \leqslant k \int_{\mathbb{D}(z_0,r)} |\operatorname{Im}\big(\partial_z f\big)| \leqslant \frac{k}{1-k} \int_{\mathbb{D}(z_0,r)} |u_y| = \mathcal{O}(r^N),$$

for almost every $z_0 \in E$ and for all positive integers N, when $r < r_0(z_0, N)$.

Suppose $w \in \mathbb{D}(z_0, r_0)$. We begin by showing that for all positive integers n

$$f(w) = \sum_{j=0}^{n-1} c_j \, (w - z_0)^j + \mathcal{E}(w), \qquad \int_{\mathbb{D}(z_0,r)} |D\mathcal{E}| dm = \mathcal{O}(r^{n+1}),$$

where $0 < r \leqslant r_0$ and $c_j \in \mathbb{C}$ are constants depending only on f and z_0.

Smoothness at a point has been studied, for example, in (Dyn′kin, 1998) and we use a few similar ideas.

Generalized Cauchy Formula (Series Representation (cont.))

The *generalized Cauchy formula* gives

$$f(w) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}(z_0,r_0)} \frac{f(z)}{z-w}dz + \frac{1}{\pi}\int_{\mathbb{D}(z_0,r_0)} \frac{\partial_{\bar z} f(z)}{w-z} dm(z), \quad w \in \mathbb{D}(z_0, r_0).$$

The first term is analytic in the disk $\mathbb{D}(z_0, r_0)$, thus

$$\sum_{j=0}^{n-1} a_j\,(w-z_0)^j + R_n(w), \qquad R_n(w) = \mathcal{O}(|w-z_0|^n).$$

The second term
$$= -\sum_{j=0}^{n-1}(w-z_0)^j\,\frac{1}{\pi}\int_{\mathbb{D}(z_0,r_0)} \frac{\partial_{\bar z} f(z)}{(z-z_0)^{j+1}}dm(z) + (w-z_0)^n\,\frac{1}{\pi}\int_{\mathbb{D}(z_0,r_0)} \frac{\partial_{\bar z} f(z)}{(z-z_0)^n(w-z)}dm(z).$$

The coefficient integrals converge: divide in annuli and use the fact that z_0 is a zero of infinite order (set $N = n+2$).

Remainder Term (Series Representation (cont.))

To show:

$$\int_{\mathbb{D}(z_0,r)} |D\mathcal{E}|\, dm = \mathcal{O}(r^{n+1}).$$

Note $\mathcal{E} = R_n + T$, where R_n is holomorphic with $R_n(w) = \mathcal{O}(|z-w|^n)$ and

$$T(w) := (w - z_0)^n \frac{1}{\pi} \int_{\mathbb{D}(z_0,r_0)} \frac{\partial_{\bar{z}} f(z)}{(z-z_0)^n (w-z)} dm(z).$$

Only the estimation of $\partial_z T$ remains.

Key points

- Higher integrability for $\partial_{\bar{z}} f$ (Astala, 1994)
- Integral term in T is a Cauchy transform of a L^p-function with a compact support and $p > 2$.

No Higher-Order Terms (Series Representation (cont.))

We have

$$f(w) = \sum_{j=0}^{n-1} c_j \, (w - z_0)^j + \mathcal{E}(w), \qquad \int_{\mathbb{D}(z_0,r)} |D\mathcal{E}| dm = \mathcal{O}(r^{n+1}).$$

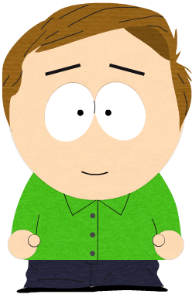
The goal was: For a.e. $z_0 \in E$, $f(w) = c_0 + c_1 \, (w - z_0) + \mathcal{E}(w)$ near the point z_0, where $c_0 \in \mathbb{C}$, $c_1 \in \mathbb{R}$ are constants depending only on f and z_0 and

$$\int_{\mathbb{D}(z_0,r)} |D\mathcal{E}| dm = \mathcal{O}(r^{n+1})$$

holds for small enough $r > 0$ and for all positive integers n.

Take $\mathrm{Im}(\partial_z \, \cdot)$. The goal follows by convergence rates of $\int_{\mathbb{D}(z_0,r)} |D\mathcal{E}| dm$ and $\int_{\mathbb{D}(z_0,r)} |\mathrm{Im}(\partial_z f)| dm$.

Thank You!



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