Kvanttilaskenta - 1. tehtävät

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1 edX-tehtävät

Vastauksissa on käytetty edX-kurssin materiaalia.

1.1 Problem 1

$$|0\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

False, sillä $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

1.2 Problem 2

$$|1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

False, sillä $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1.3 Problem 3

A quantum state is a unit vector in a complex vector space.

True

1.4 Problem 4

Measurements can only be performed in the computational (standard) basis.

False

1.5 Problem 5

The probability amplitude of $|x\rangle$ is equal to the probability that the outcome of a measurement is x.

False. Sen itseisarvon neliö sen sijaan on.

1.6 Problem 6

The inner product of $|+\rangle$ and $|-\rangle$ is 1.

False. Koska $|+\rangle$ ja $|-\rangle$ ovat ortogonaalisia, on niiden sisätulon oltava 0.

$$\begin{aligned} \langle + | - \rangle &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{1}{2} - \frac{1}{2} = 0 \end{aligned}$$

1.7 Problem 7

In $\mathbb{C}^2$, how many real unit vectors are there whose projection onto $|1\rangle$ has length $\frac{\sqrt{3}}{2}$?

4. Tämä voidaan päätellä symmetrialla ilman itse vektoreiden laskemista. Vektorit ovat $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$, $\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$, $-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ ja $-\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$.

1.8 Problem 8

In $\mathbb{C}^2$, how many complex unit vectors are there whose projection onto $|1\rangle$ has length $\frac{\sqrt{3}}{2}$?

Infinite. Näitä vektoreita on rajaton määrä, esim $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}e^{\theta i}|1\rangle$ kaikille thetan arvoille, jotka ovat jaollisia piillä.

1.9 Problem 9 (bonus)

In the double-slit experiment, consider the point at the middle of the final (detector) screen which is equidistant from the two slits. Suppose the intensity at that point is 1 when either slit is open. Now for each of the three cases (a) bullet (b) wave (c) quantum mechanics (photons or electrons) calculate the intensity at the same point when both slits are open.

Kun havaittava kohde on luoti, kasvaa intensiteetti summaamalla, eli kohdan

(a) vastaus on $1 + 1 = 2$. Aaltojen tai kvanttipartikkeleiden kohdalla intensiteetti on pisteen amplitudin neliö, ja nämä amplitudit summataan. Annetussa keskipisteessä meillä on täysin konstruktiivinen interferenssi, joten intensiteetti ja kohtien (b) ja (c) vastaus on $\left(\sqrt{1} + \sqrt{1}\right)^2 = 4$.

Kurssiassistentti ottaa mielellään vastaan ehdotuksia ylläolevan “suomentamiseksi”.

1.10 Problem 10

Suppose we have a qubit in the state $|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$. If we measure this qubit in $|u\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$, $|u^{\perp}\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ basis, what is the probability that the outcome is u?

$\frac{3}{4}$. Todennäköisyys u:lle on $\cos^2\theta$, jossa θ on $|\psi\rangle$:n ja $|u\rangle$:n välinen kulma. Tehtävässä amplitudit oli annettu niin, että $\theta = 30°$.

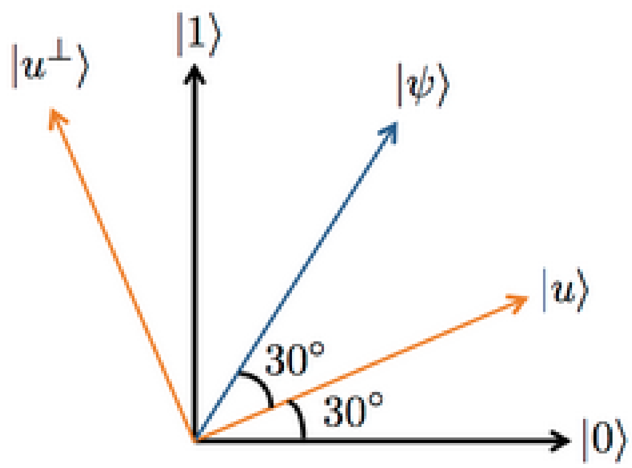

Figure 1: Tehtävä 10 kannat graafisesti esitettynä

1.11 Problem 11

Let $|\phi\rangle = a|0\rangle + b|1\rangle$ where a and b are nonnegative real numbers. We know that if we measure $|\phi\rangle$ in the standard basis, the probability of getting a 0 is $\frac{9}{25}$. What is $|\phi\rangle$?

$\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$.

$a^2 = \frac{9}{25}$ ja a:n on oltava positiivinen reaaliluku, niinpä $a = \frac{3}{5}$. Normalisoinnilla $b^2 = \frac{16}{25}$ eli $b = \frac{4}{5}$.

1.12 Problem 12

We have a qubit in the state $|\phi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$, which we want to measure in the $\{\cos\theta|0\rangle + \sin\theta|1\rangle, \sin\theta|0\rangle - \cos\theta|1\rangle\}$ basis. In order for the two possible outcomes to be equiprobable, what should be the value of θ in degrees? (Answer between 0 and 90.)

Jotta tulokset olisivat yhtä todennäköisiä, on mittauskannan oltava 45 asteen kulmassa $|\phi\rangle$:n kanssa. Koska $|\phi\rangle$ on 30 asteen kulmassa $|0\rangle$:n kanssa, saadaa tämä aikaan kun $\theta = 30° + 45° = 75°$.

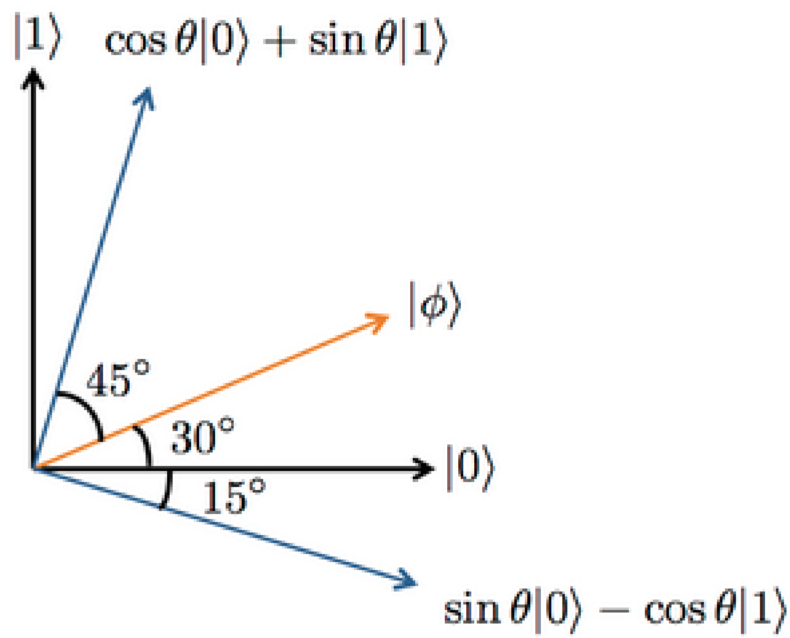

Figure 2: Tehtävä 12 kannat graafisesti esitettynä

1.13 Problem 13

(a)

A vertically polarized photon goes through two polarizing filters, the first of which is vertically aligned and the second at 45 degrees. What is the probability that the photon is transmitted through both filters?

$\frac{1}{2}$. Fotoni pääsee ensimmäisen suodattimen läpi todennäköisyydellä 1, jonka jälkeen se on edelleen pystypolarisoitu. Toisen suodattimen läpi se pääsee todennäköisyydellä $\cos^2 45° = \frac{1}{2}$.

(b)

Now, you are allowed to place a polarizing filter between the two filters in the previous question. If you wish to maximize the probability that the photon is transmitted through all three filters, what angle would you orient the additional filter? Here, assume that a 0° filter corresponds to a horizontal filter and 90° a vertical filter. Provide your answer in degrees as a real number between 0 and 90.

67.5°. Todennäköisyys, että fotoni pääsee kaikkien kolmen suodattimen läpi on suodatinkohtaisten todennäköisyyksien tulo. Kulmalla θ tämä tulo on $1 \times \cos^2(90° - \theta) \times \cos^2(\theta - 45°)$, joka saa maksimiarvonsa kun $\theta = 67.5°$.

(c)

In that case, what is the probability that the photon is transmitted through all three? Round your answer to the nearest thousandth. (ex: 0.182)

$1 \times \cos^2(90° - 67.5°) \times \cos^2(67.5° - 45°) = 0.72855\ldots$

1.14 Problem 14

A qubit is either in the state $|u\rangle = \cos\left(\frac{\pi}{4} - \frac{x}{2}\right)|0\rangle + \sin\left(\frac{\pi}{4} - \frac{x}{2}\right)|1\rangle$ or $|v\rangle = \cos\left(\frac{\pi}{4} + \frac{x}{2}\right)|0\rangle + \sin\left(\frac{\pi}{4} + \frac{x}{2}\right)|1\rangle$ and we want to determine which state it is in by measuring it. One of the following two measurements is optimal in terms of the probability of success.

Measurement I: Measure in the basis $|u\rangle = \cos\left(\frac{\pi}{4} - \frac{x}{2}\right)|0\rangle + \sin\left(\frac{\pi}{4} - \frac{x}{2}\right)|1\rangle$, $|u^\perp\rangle = -\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)|0\rangle + \cos\left(\frac{\pi}{4} - \frac{x}{2}\right)|1\rangle$. If the outcome is u, guess that the qubit was in the state $|u\rangle$, and if the outcome was $u^\perp$, guess that it was in state $|v\rangle$.

Measurement II: Measure in the standard basis and interpret 0 as $|u\rangle$ and 1 as $|v\rangle$.

The probability of success is defined as $\frac{1}{2}p_u + \frac{1}{2}p_v$, where $p_u = Pr[$we guess $|u\rangle$ given that the qubit was in the state $|u\rangle]$ and $p_v = Pr[$we guess $|v\rangle$ given that the qubit was in the state $|v\rangle]$.

(a)

What is the probability of success of Measurement I as a function of x?

$\frac{1+\sin^2 x}{2}$. $p_u = 1$ ja $p_v = sin^2(|u\rangle$:n ja $|v\rangle$:n välinen kulma$) = sin^2 x$. Niinpä $\frac{1}{2}p_u + \frac{1}{2}p_v = \frac{1+\sin^2 x}{2}$.

(b)

$\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$.

Tässä tapauksessa sekä $|u\rangle$:n ja $|0\rangle$:n että $|v\rangle$:n ja $|1\rangle$:n välinen kulma on $\frac{\pi}{4}-\frac{x}{2}$. Täten $p_u = p_v = cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$ ja $\frac{1}{2}p_u + \frac{1}{2}p_v = sin^2\frac{x}{2} = cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$

(c)

Now note that the following becomes good approximations as $\theta \to 0$:

$$\sin\theta \approx \theta$$
$$\sin^2\left(\frac{\pi}{4}-\theta\right) \approx \frac{1}{2}-\theta$$

Use these approximations to estimate the probability of success of the two measurements as $x \to 0$ as a function of x.

Measurement I: $\frac{1+x^2}{2}$ sillä $\frac{1+\sin^2 x}{2} \approx \frac{1+x^2}{2}$
Measurement II: $\frac{1+x}{2}$ sillä $cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right) = 1-\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right) \approx \frac{1}{2}+\frac{x}{2} = \frac{1+x}{2}$

(d)

Based on your answer to part (c), which measurement better distinguishes between $|u\rangle$ and $|v\rangle$?

Measurement II. Jos x on lähellä 0:aa $\frac{1+x^2}{2} < \frac{1+x}{2}$

2 QCE tehtävät

2.1

A quantum system is in the state

$$\frac{1-i}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

If a measurement is made, what is the probability the system is in state $|0\rangle$ or in state $|1\rangle$

Todennäköisyys on kannan amplitudin itseisarvon neliö eli $|0\rangle$:lle

$$\begin{aligned}\left|\frac{1-i}{\sqrt{3}}\right|^2 &= \left(\frac{1-i}{\sqrt{3}}\right)^*\left(\frac{1-i}{\sqrt{3}}\right)\\ &= \left(\frac{1+i}{\sqrt{3}}\right)\left(\frac{1-i}{\sqrt{3}}\right)\\ &= \frac{1-i+i+1}{3} = \frac{2}{3}\end{aligned}$$

ja $|1\rangle$:lle

$$\left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3} = 1 - \frac{2}{3}$$

2.2

Two quantum states are given by

$$|a\rangle = \begin{pmatrix} \text{-4i} \\ 2 \end{pmatrix}, \quad |b\rangle = \begin{pmatrix} 1 \\ \text{-1+i} \end{pmatrix}$$

(A)

Find $|a+b\rangle$

$$\begin{aligned} |a+b\rangle &= \begin{pmatrix} \text{-4i} \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ \text{-1+i} \end{pmatrix} \\ &= \begin{pmatrix} \text{1-4i} \\ \text{1+i} \end{pmatrix} \end{aligned}$$

(B)

Calculate $3\,|a\rangle - 2\,|b\rangle$

$$\begin{aligned} 3\,|a\rangle - 2\,|b\rangle &= 3\begin{pmatrix} \text{-4i} \\ 2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ \text{-1+i} \end{pmatrix} \\ &= \begin{pmatrix} \text{-12i} \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ \text{-2+2i} \end{pmatrix} \\ &= \begin{pmatrix} \text{-2-12i} \\ \text{8-2i} \end{pmatrix} \end{aligned}$$

(C)

Normalize $|a\rangle$, $|b\rangle$

$$\begin{aligned} \langle a| &= (|a\rangle)^\dagger = (|a\rangle^T)^* = \begin{pmatrix} \text{4i} & 2 \end{pmatrix} \\ \langle a|a\rangle &= \begin{pmatrix} \text{4i} & 2 \end{pmatrix}\begin{pmatrix} \text{-4i} \\ 2 \end{pmatrix} \\ &= 4i(-4i) + 2 \times 2 = 20 \end{aligned}$$

Niinpä $\|a\| = \sqrt{20}$ ja $|\tilde{a}\rangle = \frac{1}{\sqrt{20}}\,|a\rangle$
Sama $|b\rangle$:lle

$$\begin{aligned} \langle b| &= \begin{pmatrix} 1 & \text{-1-i} \end{pmatrix} \\ \langle b|b\rangle &= \begin{pmatrix} 1 & \text{-1-i} \end{pmatrix}\begin{pmatrix} 1 \\ \text{-1+i} \end{pmatrix} \\ &= 1 \times 1 + (-1-i)(-1+i) = 3 \end{aligned}$$

joten $\|b\| = \sqrt{3}$ ja $|\tilde{b}\rangle = \frac{1}{\sqrt{3}} |b\rangle$

2.3

Another basis for $\mathbb{C}^2$ is

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Invert this relation to express $\{|0\rangle, |1\rangle\}$ in terms of $\{|+\rangle, |-\rangle\}$

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \quad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

2.4

A quantum system is in the state

$$|\psi\rangle = \frac{3i\,|0\rangle + 4\,|1\rangle}{5}$$

(A)

Is the state normalized?

Vektori ψ on normalisoitu kun $\langle\psi|\psi\rangle = 1$

$$\begin{aligned}\langle\psi|\psi\rangle &= \frac{1}{5^2}\begin{pmatrix} \text{-3i} & 4 \end{pmatrix}\begin{pmatrix} \text{3i} \\ 4 \end{pmatrix} \\ &= \frac{9+16}{25} = 1\end{aligned}$$

joten tila on normalisoitu.

(B)

Express the state in the $|+\rangle, |-\rangle$ basis.

Tehtävän 2.3 mukaan

$$\begin{aligned}|\psi\rangle &= \frac{3i}{5}\,\frac{|+\rangle + |-\rangle}{\sqrt{2}} + \frac{4}{5}\frac{|+\rangle - |-\rangle}{\sqrt{2}} \\ &= \frac{1}{5}\left(\frac{4+3i}{\sqrt{2}}\,|+\rangle + \frac{-4+3i}{\sqrt{2}}\,|-\rangle\right) \\ &= \frac{4+3i}{5\sqrt{2}}\,|+\rangle + \frac{-4+3i}{5\sqrt{2}}\,|-\rangle\end{aligned}$$

2.5

Use the Gram-Schmidt process to find an orthonormal basis for a subspace of the four-dimensional space $\mathbb{R}^4$ spanned by

$$|u_1\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad |u_2\rangle = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix}, \quad |u_3\rangle = \begin{pmatrix} 1 \\ -3 \\ -4 \\ -2 \end{pmatrix}$$

Ensimmäinen kantavektori on $|\tilde{w}_1\rangle = |u_1\rangle$, normalisoidaan se:

$$\langle \tilde{w}_1 | \tilde{w}_1 \rangle = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 4$$

$$|w_1\rangle = \frac{\tilde{w}_1}{\sqrt{\langle u_1 | u_1 \rangle}} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Toinen vektori on $|\tilde{w}_2\rangle = |u_2\rangle - \frac{\langle \tilde{w}_1 | u_2 \rangle}{\langle \tilde{w}_1 | \tilde{w}_1 \rangle} |\tilde{w}_1\rangle$

$$\langle \tilde{w}_1 | u_2 \rangle = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix} = 12$$

$$|\tilde{w}_2\rangle = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\langle \tilde{w}_2 | \tilde{w}_2 \rangle = \begin{pmatrix} -2 & -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix} = 10$$

$$|w_2\rangle = \frac{1}{\sqrt{10}} \begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix}$$

Tarkistetaan ortogoniaalius

$$\langle w_1|w_2\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \times \frac{1}{\sqrt{10}}\begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix} = 0$$

Ja kolmas vektori on $|\tilde{w}_3\rangle = |u_3\rangle - \frac{\langle\tilde{w}_1|u_3\rangle}{\langle\tilde{w}_1|\tilde{w}_1\rangle}|\tilde{w}_1\rangle - \frac{\langle\tilde{w}_2|u_3\rangle}{\langle\tilde{w}_2|\tilde{w}_2\rangle}|\tilde{w}_2\rangle$

$$\langle\tilde{w}_1|u_3\rangle = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ -4 \\ -2 \end{pmatrix} = -8$$

$$\langle\tilde{w}_2|u_3\rangle = \begin{pmatrix} -2 & -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ -4 \\ -2 \end{pmatrix} = -7$$

$$|\tilde{w}_3\rangle = \begin{pmatrix} 1 \\ -3 \\ -4 \\ -2 \end{pmatrix} - \frac{-8}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-7}{10}\begin{pmatrix} -2 \\ -1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 16 \\ -17 \\ -13 \\ 14 \end{pmatrix}$$

$$\langle\tilde{w}_3|\tilde{w}_3\rangle = \frac{1}{10}\begin{pmatrix} 16 & -17 & -13 & 14 \end{pmatrix}\frac{1}{10}\begin{pmatrix} 16 \\ -17 \\ -13 \\ 14 \end{pmatrix} = \frac{91}{10}$$

$$|w_3\rangle = \frac{1}{\sqrt{910}}\begin{pmatrix} 16 \\ -17 \\ -13 \\ 14 \end{pmatrix}$$

Tarkistetaan ortogonaalisuus:

$$\langle w_1|w_3\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \times \frac{1}{\sqrt{910}}\begin{pmatrix} 16 \\ -17 \\ -13 \\ 14 \end{pmatrix} = 0$$

$$\langle w_2|w_3\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} -2 & -1 & 1 & 2 \end{pmatrix} \times \frac{1}{\sqrt{910}}\begin{pmatrix} 16 \\ -17 \\ -13 \\ 14 \end{pmatrix} = 0$$

2.6

Photon horizontal and vertical polarizations are written as $|h\rangle$ and $|v\rangle$, respectively. Suppose

$$\begin{aligned}
|\phi_1\rangle &= \frac{1}{2}|h\rangle + \frac{\sqrt{3}}{2}|v\rangle \\
|\phi_2\rangle &= \frac{1}{2}|h\rangle - \frac{\sqrt{3}}{2}|v\rangle \\
|\phi_3\rangle &= |h\rangle
\end{aligned}$$

Find

$$|\langle\phi_1|\phi_2\rangle|^2, \quad |\langle\phi_1|\phi_3\rangle|^2, \quad |\langle\phi_3|\phi_2\rangle|^2$$

$$\begin{aligned}
|\langle\phi_1|\phi_2\rangle|^2 &= \left|\frac{1}{2}\times\frac{1}{2}+\frac{\sqrt{3}}{2}\times\frac{-\sqrt{3}}{2}\right|^2 \\
&= \left|\frac{1}{4}-\frac{3}{4}\right|^2 \\
&= \left|-\frac{1}{2}\right|^2 = \frac{1}{4} \\
|\langle\phi_1|\phi_3\rangle|^2 &= \left|\frac{1}{2}\times 1+\frac{\sqrt{3}}{2}\times 0\right|^2 \\
&= \left|\frac{1}{2}\right|^2 = \frac{1}{4} \\
|\langle\phi_3|\phi_2\rangle|^2 &= \left|1\times\frac{1}{2}+0\times\frac{-\sqrt{3}}{2}\right|^2 \\
&= \left|\frac{1}{2}\right|^2 = \frac{1}{4}
\end{aligned}$$