Kvanttilaskenta - 2. tehtävät

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1 edX-tehtävät

Vastauksissa on käytetty edX-kurssin materiaalia.

1.1 Problem 1

The inner product of $|+\rangle$ and $|-\rangle$ is 1.

Edelleen false, kts. viikon 1 tehtävä 6.

1.2 Problem 2

The standard basis for a two-qubit quantum system is $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

True.

1.3 Problem 3

Any state of a two-qubit quantum system can be written in the form $(a|0\rangle + b|1\rangle)(c|0\rangle + c|1\rangle).(a, b, c, d \in \mathbb{C})$

False. Esim Bell-tila $\Phi^+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ ei voida kirjoittaa yo. muodossa (kts. Course Notes, 2.1)

1.4 Problem 4

Two qubits are entangled if their state can be written as $(a|0\rangle + b|1\rangle)(a|0\rangle + b|1\rangle)$. $(a, b \in \mathbb{C})$

False. Kts. edellinen tehtävä

1.5 Problem 5

$a|00\rangle + b|11\rangle = a|{+}{+}\rangle + |{-}{-}\rangle$ for any $a, b \in \mathbb{C}$ that satisfy the normalization condition.

False. Esim $a = 0, b = 1$.

1.6 Problem 6

Bell's theorem implies that there is no local hidden variable theory (local realism) that is consistent with the predictions of quantum mechanics.

True

1.7 Problem 7

What is the inner product of $\frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle$ and $|+\rangle$?

Ottaen huomioon, että tiedämme edellisviikolta $\langle u|u\rangle = 1$ ja $\langle u|u^{\perp}\rangle = 0$.

$$\begin{aligned}
\langle +|\left(\frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle\right) &= \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)\left(\frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle\right) \\
&= \frac{1}{5\sqrt{2}}\left(3\langle 0|0\rangle - 4\langle 0|1\rangle + 3\langle 1|0\rangle - 4\langle 1|1\rangle\right) \\
&= -\frac{1}{5\sqrt{2}}
\end{aligned}$$

1.8 Problem 8

Rewrite $\frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle$ in the sign basis.

$$\begin{aligned}
\frac{3}{5}|0\rangle - \frac{4}{5}|1\rangle &= \frac{3}{5}\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) - \frac{4}{5}\left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right) \\
&= -\frac{1}{5\sqrt{2}}|+\rangle + \frac{7}{5\sqrt{2}}|-\rangle
\end{aligned}$$

1.9 Problem 9

Suppose that a qubit is in the state $|\phi\rangle = a|0\rangle + \sqrt{1-a^2}|1\rangle$ where $a \in [-1, 1]$. If we first perform a standard basis measurement on this qubit and then perform a $|u\rangle, |u^{\perp}\rangle$-basis measurement where $|u\rangle = b|0\rangle + \sqrt{1-b^2}|1\rangle$ for some $b \in [-1, 1]$, what is the probability that the outcome of the second measurement is u, in terms of a and b?

Merkitään ensimmäisen ja toisen mittauksen tulos x_1 ja x_2

$$\begin{aligned} P(x_2 = u) &= P(x_1 = 0)P(x_2 = u \mid x_1 = 0) + P(x_1 = 1)P(x_2 = u \mid x_1 = 1) \\ &= a^2b^2 + (1 - a^2)(1 - b^2) \end{aligned}$$

1.10 Problem 10

Pick one of the following four alternatives which is not an orthogonal basis for a two-qubit system, or options 5 or 6 if none or all are valid.

3. vaihtoehto $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$, $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$, $\frac{1}{\sqrt{2}}|{+}{+}\rangle + \frac{1}{\sqrt{2}}|{-}{-}\rangle$, $\frac{1}{\sqrt{2}}|{+}{-}\rangle + \frac{1}{\sqrt{2}}|{-}{+}\rangle$ koska $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}|{+}{+}\rangle + \frac{1}{\sqrt{2}}|{-}{-}\rangle$ eli eivät ole ortogonaalisia.

1.11 Problem 11

If the first qubit is in the state $\frac{\sqrt{2}}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$ and the second qubit is in the state $\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle$, what is the state of the composite system?

$$\frac{\sqrt{2}}{3}|00\rangle + \frac{2}{3}|01\rangle + \frac{1}{3}|10\rangle + \frac{\sqrt{2}}{3}|11\rangle$$

1.12 Problem 12

Factor $\frac{1}{2\sqrt{2}}|00\rangle - \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle$, where $|a|^2 + |b|^2 = 1$ and $|c|^2 + |d|^2 = 1$. What is the value of $|a|$?

$\frac{1}{2}$. Käytetään $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$, jossa tehtävän mukaan siis $ac = \frac{1}{2\sqrt{2}}, ad = -\frac{1}{2\sqrt{2}}, bc = \frac{\sqrt{3}}{2\sqrt{2}}$ ja $bd = -\frac{\sqrt{3}}{2\sqrt{2}}$. Koska $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$ on oltava $a = \frac{1}{2}, c = \frac{1}{\sqrt{2}}, d = -\frac{1}{\sqrt{2}}$ ja siis $b = \frac{\sqrt{3}}{2}$.

1.13 Problem 13

Suppose we have two qubits in the state $\alpha|00\rangle + \beta|11\rangle$.

(a)

If we measure the first qubit in the sign basis, what is the probability of seeing a +?

$\frac{1}{2}$.

$$\begin{aligned}
\alpha |0\rangle |0\rangle + \beta |1\rangle |1\rangle &= \alpha \left(\frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \right) |0\rangle + \beta \left(\frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \right) |1\rangle \\
&= \frac{\alpha}{\sqrt{2}} |+0\rangle + \frac{\alpha}{\sqrt{2}} |-0\rangle + \frac{\beta}{\sqrt{2}} |+1\rangle - \frac{\beta}{\sqrt{2}} |-1\rangle \\
P(+) &= \left(\frac{\alpha}{\sqrt{2}} \right)^2 + \left(\frac{\beta}{\sqrt{2}} \right)^2 \\
&= \frac{\alpha^2 + \beta^2}{2} = \frac{1}{2}
\end{aligned}$$

(b)

What is the resulting state of the second qubit in that case?

Mikäli + on havaittu, on tila siis $\frac{\alpha}{\sqrt{2}} |0\rangle + \frac{\beta}{\sqrt{2}} |1\rangle$, uudelleennormalisoituna $\alpha |0\rangle + \beta |1\rangle$

1.14 Problem 14

A two-qubit system was originally in the state $\frac{1}{5} |00\rangle + \frac{2}{5} |01\rangle + \frac{4}{5} |10\rangle - \frac{2}{5} |11\rangle$, and then we measured the first qubit to be 0. Now, if we measure the second qubit in the standard basis, what is the probability that the outcome is 0?

Mittauksen jälkeen on tila siis $|0\rangle \left(\frac{1}{5} |0\rangle + \frac{2}{5} |1\rangle \right)$. Uudelleennormalisoituna tämä on $|0\rangle \left(\frac{1}{\sqrt{5}} |0\rangle + \frac{2}{\sqrt{5}} |1\rangle \right)$. Siispä todennäköisyys 0:lle toisella mittauksella on $| \frac{1}{\sqrt{5}} |^2 = \frac{1}{5}$

1.15 Problem 15

We have two qubits originally in the state $|0+\rangle$ (which means that the first qubit is in $|0\rangle$ and the second qubit is in $|0\rangle$), which we want to entangle by performing a measurement. Which of the following measurements should we perform?

$\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$, $\frac{1}{\sqrt{2}} |00\rangle - \frac{1}{\sqrt{2}} |11\rangle$, $\frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle$, $\frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |10\rangle$ on ainoa vaihtoehto, jossa kaikki perusvektorit ovat lomittuneita.

1.16 Problem 16

Again we have two qubits in the state $|0+\rangle$, but this time we want to entangle them by performing a partial measurement on the first qubit. Which of the following measurements should we perform on the first qubit?

Ensimmäisen kubitin osittaismittauksella ei voida saada aikaan lomittunutta tilaa. Jos kubitti mitattaisiin vapaavalinteisessa kannassa u, olisi järjestelmän tila sen jälkeen $|u\rangle|+\rangle$, joka ei ole lomittunut.

1.17 Problem 17

How do we write $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$ in the sign basis?

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) + \frac{e^{i\theta}}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right) \\
&= \frac{1+e^{i\theta}}{2}|+\rangle + \frac{1-e^{i\theta}}{2}|-\rangle
\end{aligned}
$$

1.18 Problem 18

Consider the state $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$ from the previous question. To estimate the phase θ, we measure $|psi\rangle$ in the sign basis. What is the probability that the outcome of the measurement is $+$?

$$
\begin{aligned}
P(+) &= \left|\frac{1+e^{i\theta}}{2}\right|^2 \\
&= \left(\frac{1+e^{i\theta}}{2}\right)\left(\frac{1+e^{-i\theta}}{2}\right) \\
&= \frac{1}{4}\left(1+e^{-i\theta}+e^{i\theta}+1\right) \\
&= \frac{2+\cos(-\theta)+i\sin(-\theta)+\cos\theta+i\sin\theta}{4} \\
&= \frac{1+\cos\theta}{2}
\end{aligned}
$$

sillä $\cos(-\theta) = \cos\theta$ ja $\sin(-\theta) = -\sin\theta$.

2 QCE-Tehtävät

3.1

Verify that the outer product representations of X and Y are given by $X = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ by letting them act on the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and comparing with (3.9) and (3.10).

$$
\begin{aligned}
X|\psi\rangle &= \alpha\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)|0\rangle + \beta\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)|1\rangle \\
&= \alpha\left(|0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle\right) + \beta\left(|0\rangle\langle 1|1\rangle + |1\rangle\langle 0|1\rangle\right) \\
&= \alpha|1\rangle + \beta|0\rangle
\end{aligned}
$$

Kun käytämme $\langle 0|1\rangle = \langle 1|0\rangle = 0$ ja $\langle 0|0\rangle = \langle 1|1\rangle = 1$. Tulos on yhteneväinen (3.9):n kanssa, eli $X = \sigma_1$

$$\begin{aligned} Y|\psi\rangle &= \alpha\left(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\right)|0\rangle + \beta\left(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\right)|1\rangle \\ &= \alpha\left(-i|0\rangle\langle 1|0\rangle + i|1\rangle\langle 0|0\rangle\right) + \beta\left(-i|0\rangle\langle 1|1\rangle + i|1\rangle\langle 0|1\rangle\right) \\ &= \alpha i|1\rangle - \beta i|0\rangle \end{aligned}$$

Tästä huomaamme, että kirjassa on painovirhe kaavassa 3.10 (joko negaatiossa tai ketien järjestyksessä).

3.2

Show that the matrix representation of the X operator with respect to the computational basis is

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Käytetään jälleen $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$\begin{aligned} X|\psi\rangle &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \end{aligned}$$

3.3

Consider the basis states given by

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Show that the matrix representation of the X operator with respect to this basis is

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Käytetään nyt $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$, joka on standardikannassa $\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$

$$\begin{aligned} X|\psi\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \alpha|+\rangle - \beta|-\rangle \\ &= \alpha\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) - \beta\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \\ &= \frac{\alpha-\beta}{\sqrt{2}}|0\rangle + \frac{\alpha+\beta}{\sqrt{2}}|1\rangle \end{aligned}$$

3.4

Consider the space $\mathbb{C}^3$ with the basis $\{|1\rangle, |2\rangle, |3\rangle\}$. An operator $\hat{A}$ is given by $\hat{A} = i|1\rangle\langle 1| + \frac{\sqrt{3}}{2}|1\rangle\langle 2| + 2|2\rangle\langle 1| - |2\rangle\langle 3|$. Write down the adjoint of this operator $\hat{A}^\dagger$

$$\begin{aligned}\hat{A}^\dagger &= (i|1\rangle\langle 1|)^\dagger + \left(\frac{\sqrt{3}}{2}|1\rangle\langle 2|\right)^\dagger + (2|2\rangle\langle 1|)^\dagger - (|2\rangle\langle 3|)^\dagger \\ &= -i|1\rangle\langle 1| + \frac{\sqrt{3}}{2}|2\rangle\langle 1| + 2|1\rangle\langle 2| - |3\rangle\langle 2|\end{aligned}$$

3.5

Find the eigenvalues and eigenvectors of the X operator.

Lasketaan ensin ominaisarvot:

$$\begin{aligned}X - \lambda I &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \\ \det|X - \lambda I| &= \det\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (-\lambda)^2 - 1^2 = \lambda^2 - 1 \\ \lambda_{1,2} &= \pm 1\end{aligned}$$

$\lambda_1 = 1$, joten sen yhtälö on $X|\phi_1\rangle = |\phi_1\rangle$

$$\begin{aligned}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} a \\ b \end{pmatrix} \\ \begin{pmatrix} b \\ a \end{pmatrix} &= \begin{pmatrix} a \\ b \end{pmatrix} \\ &\begin{cases} b - a = 0 \\ a - b = 0 \end{cases} \\ |\phi_1\rangle &= \begin{pmatrix} a \\ a \end{pmatrix}\end{aligned}$$

Koska $\langle\phi_1|\phi_1\rangle = 1$ ja $\langle\phi_1| = \begin{pmatrix} a^* & a^* \end{pmatrix}$ saamme $1 = \langle\phi_1|\phi_1\rangle = \begin{pmatrix} a^* & a^* \end{pmatrix}\begin{pmatrix} a \\ a \end{pmatrix} = 2|a|^2 \Rightarrow a = \frac{1}{\sqrt{2}}$

$\lambda_2 = -1$, vastaavasti $X|\phi_2\rangle = -|\phi_2\rangle$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$
$$\begin{cases} a+b=0 \\ a+b=0 \end{cases}$$
$$|\phi_2\rangle = \begin{pmatrix} a \\ -a \end{pmatrix}$$

Jälleen $\langle\phi_2|\phi_2\rangle = 1$ ja $\langle\phi_2| = \begin{pmatrix} a^* & -a^* \end{pmatrix}$ joten $1 = \langle\phi_1|\phi_1\rangle = \begin{pmatrix} a^* & -a^* \end{pmatrix} \begin{pmatrix} a \\ -a \end{pmatrix} = 2|a|^2 \Rightarrow a = \frac{1}{\sqrt{2}}$. Ominaisvektorit ovat siis $|\phi_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ ja $|\phi_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Tarkistetaan vielä

$$X|\phi_1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = |\phi_1\rangle$$
$$X|\phi_2\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = -|\phi_2\rangle$$

3.6

Show that the Y operator is traceless

$$Tr(Y) = Tr \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 0$$

3.7

Find the eigenvalues of

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$$

$$B - \lambda I = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 3-\lambda & 4 \\ 1 & 0 & 2-\lambda \end{pmatrix}$$
$$\det|B - \lambda I| = \det\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 3-\lambda & 4 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$
$$= ((1-\lambda)(3-\lambda)(2-\lambda) + 0 + 0) - (2(3-\lambda) + 0 + 0) = -((\lambda - 3)^2\lambda)$$

Josta nähdään, että ominaisarvot ovat 0 ja 3.

3.8

Prove the following relations involving the trace operation:

$$Tr(A+B) = Tr(A) + Tr(B)$$
$$Tr(\lambda A) = \lambda Tr(A)$$
$$Tr(AB) = Tr(BA)$$

Olkoon A ja B $n \times n$-kokoisia matriiseja. Tällöin A:n jälki on $\sum_{i=1}^{n} a_{ii}$ ja siis

$$\begin{aligned}
Tr(A+B) &= \sum_{i=1}^{n}(a_{ii} + b_{ii}) \\
&= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\
&= Tr(A) + Tr(B) \\
Tr(\lambda A) &= \sum_{i=1}^{n} \lambda a_{ii} \\
&= \lambda \sum_{i=1}^{n} a_{ii} \\
&= \lambda Tr(A) \\
Tr(AB) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} \\
&= Tr(BA)
\end{aligned}$$

3.9

Show that $X = |0\rangle\langle 1| + |1\rangle\langle 0| = P_+ - P_-$

$$\begin{aligned}
P_+ - P_- &= |+\rangle\langle +| - |-\rangle\langle -| \\
&= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) - \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \\
&= |0\rangle\langle 1| + |1\rangle\langle 0| = X
\end{aligned}$$

3.10

A three-state system is in the state

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|2\rangle$$

Write down the necessary projection operators and calculate the probabilities Pr(0), Pr(1), and Pr(2).

$$\begin{aligned}
P_0 = |0\rangle\langle 0| &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
P_1 = |1\rangle\langle 1| &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
P_2 = |2\rangle\langle 2| &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
P_0|\psi\rangle &= (|0\rangle\langle 0|)\left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|2\rangle\right) \\
&= \frac{1}{2}|0\rangle\langle 0|0\rangle + \frac{1}{2}|0\rangle\langle 0|1\rangle - \frac{i}{\sqrt{2}}|0\rangle\langle 0|2\rangle \\
&= \frac{1}{2}|0\rangle \\
Pr(0) &= \langle\psi|P_0|\psi\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2}\langle 0| + \frac{1}{2}\langle 1| + \frac{i}{\sqrt{2}}\langle 2|\right)\frac{1}{2}|0\rangle \\
&= \frac{1}{4}\langle 0|0\rangle + \frac{1}{4}\langle 1|0\rangle + \frac{i}{2\sqrt{2}}\langle 2|0\rangle \\
&= \frac{1}{4} \\
P_1|\psi\rangle &= (|1\rangle\langle 1|)\left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|2\rangle\right) \\
&= \frac{1}{2}|1\rangle\langle 1|0\rangle + \frac{1}{2}|1\rangle\langle 1|1\rangle - \frac{i}{\sqrt{2}}|1\rangle\langle 1|2\rangle \\
&= \frac{1}{2}|1\rangle \\
Pr(1) &= \langle\psi| P_1 |\psi\rangle \\
&= \left(\frac{1}{2}\langle 0| + \frac{1}{2}\langle 1| + \frac{i}{\sqrt{2}}\langle 2|\right)\frac{1}{2}|1\rangle \\
&= \frac{1}{4}\langle 0|1\rangle + \frac{1}{4}\langle 1|1\rangle + \frac{i}{2\sqrt{2}}\langle 2|1\rangle \\
&= \frac{1}{4} \\
P_2|\psi\rangle &= (|2\rangle\langle 2|)\left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|2\rangle\right) \\
&= \frac{1}{2}|2\rangle\langle 2|0\rangle + \frac{1}{2}|2\rangle\langle 2|1\rangle - \frac{i}{\sqrt{2}}|2\rangle\langle 2|2\rangle \\
&= -\frac{i}{\sqrt{2}}|2\rangle \\
Pr(2) &= \langle\psi| P_2 |\psi\rangle \\
&= \left(\frac{1}{2}\langle 0| + \frac{1}{2}\langle 1| + \frac{i}{\sqrt{2}}\langle 2|\right)\left(-\frac{i}{\sqrt{2}}|2\rangle\right) \\
&= -\frac{i}{2\sqrt{2}}\langle 0|2\rangle - \frac{i}{2\sqrt{2}}\langle 1|2\rangle + \frac{1}{2}\langle 2|2\rangle \\
&= \frac{1}{2}
\end{aligned}
$$

3.11

In Example 3.17 we showed that $[\sigma_1, \sigma_2] = 2i\sigma_3$. Following the same procedure, show that $[\sigma_2, \sigma_3] = 2i\sigma_1$ and $[\sigma_3, \sigma_1] = 2i\sigma_2$.

$$
\begin{aligned}
\sigma_2\sigma_3 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
\sigma_3\sigma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}
\end{aligned}
$$

$$\sigma_2\sigma_3 - \sigma_3\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2i\sigma_1$$
$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_3\sigma_1 - \sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2i\sigma_2$$

3.12

Show that $\{\sigma_i, \sigma_j\} = 0$ when $i \neq j$.

Koska $\{A, B\} = AB + BA$ voimme käyttää edellisessä tehtävässä laskettuja arvoja ja näemme, että $\{\sigma_2, \sigma_3\} = 0_{2,2}$, samoin $\{\sigma_3, \sigma_1\}$. $\{\sigma_1, \sigma_2\}$:n arvot ovat laskettuna kirjassa (example 3.17) ja nekin pitävät.

4.1

Consider the basis in Example 4.1 Show that it is orthonormal.

Jotta kanta olisi ortonormaali, on pädettävä $\langle w_i|w_i\rangle = 1$ ja $\langle w_i|w_j\rangle = 0, i \neq j$.

$$\langle w_1|w_1\rangle = (\langle 0| \langle 0|)(|0\rangle |0\rangle) = \langle 0|0\rangle \langle 0|0\rangle = 1$$
$$\langle w_2|w_2\rangle = (\langle 0| \langle 1|)(|0\rangle |1\rangle) = \langle 0|0\rangle \langle 1|1\rangle = 1$$
$$\langle w_3|w_3\rangle = (\langle 1| \langle 0|)(|1\rangle |0\rangle) = \langle 0|0\rangle \langle 1|1\rangle = 1$$
$$\langle w_4|w_4\rangle = (\langle 1| \langle 1|)(|1\rangle |1\rangle) = \langle 1|1\rangle \langle 1|1\rangle = 1$$
$$\langle w_1|w_2\rangle = (\langle 0| \langle 0|)(|0\rangle |1\rangle) = \langle 0|0\rangle \langle 0|1\rangle = 0$$
$$\langle w_1|w_3\rangle = (\langle 0| \langle 0|)(|1\rangle |0\rangle) = \langle 0|0\rangle \langle 0|1\rangle = 0$$
$$\langle w_1|w_4\rangle = (\langle 0| \langle 0|)(|1\rangle |1\rangle) = \langle 0|1\rangle \langle 0|1\rangle = 0$$
$$\langle w_2|w_1\rangle = (\langle 0| \langle 1|)(|0\rangle |0\rangle) = \langle 0|0\rangle \langle 1|0\rangle = 0$$
$$\langle w_2|w_3\rangle = (\langle 0| \langle 1|)(|1\rangle |0\rangle) = \langle 0|1\rangle \langle 1|0\rangle = 0$$
$$\langle w_2|w_4\rangle = (\langle 0| \langle 1|)(|1\rangle |1\rangle) = \langle 0|1\rangle \langle 1|1\rangle = 0$$
$$\langle w_3|w_1\rangle = (\langle 1| \langle 0|)(|0\rangle |0\rangle) = \langle 0|0\rangle \langle 1|0\rangle = 0$$
$$\langle w_3|w_2\rangle = (\langle 1| \langle 0|)(|0\rangle |1\rangle) = \langle 0|1\rangle \langle 1|0\rangle = 0$$
$$\langle w_3|w_4\rangle = (\langle 1| \langle 0|)(|1\rangle |1\rangle) = \langle 1|0\rangle \langle 1|1\rangle = 0$$
$$\langle w_4|w_1\rangle = (\langle 1| \langle 1|)(|0\rangle |0\rangle) = \langle 1|0\rangle \langle 1|0\rangle = 0$$
$$\langle w_4|w_2\rangle = (\langle 1| \langle 1|)(|0\rangle |1\rangle) = \langle 1|0\rangle \langle 1|1\rangle = 0$$
$$\langle w_4|w_3\rangle = (\langle 1| \langle 1|)(|0\rangle |1\rangle) = \langle 1|0\rangle \langle 1|1\rangle = 0$$

Eli kanta on ortonormaali.

4.2

Returning to Example 4.1, show that $\langle w_3|w_4\rangle = \langle w_4|w_3\rangle = 0$

Kts edellinen tehtävä.

4.3

Given that $\langle a|b\rangle = 1/2$ and $\langle c|d\rangle = 3/4$, calculate $\langle\psi|\phi\rangle$, where $|\psi\rangle = |a\rangle \otimes |c\rangle$ and $|\phi\rangle = |b\rangle \otimes |d\rangle$.

$$\langle\psi|\phi\rangle = (\langle a| \otimes \langle c|)(|b\rangle \otimes |d\rangle) = \langle a|b\rangle \langle c|d\rangle = \frac{1}{2}\frac{3}{4} = \frac{3}{8}$$

4.4

Calculate the tensor product of

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |\phi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

Koska tensoritulo on lineaarinen skalaareihin nähden, lasketaan

$$\begin{aligned} |\psi\rangle \otimes |\phi\rangle &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}\right] \\ &= \left(\frac{1}{2\sqrt{2}}\right)\begin{pmatrix} 1 \\ \sqrt{3} \\ 1 \\ \sqrt{3} \end{pmatrix} \end{aligned}$$

4.5

Can $|\psi\rangle = \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle)$ be written as a product state?

Kyllä voidaan.

$$\begin{aligned} |\psi\rangle &= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \\ &= \left(\frac{1}{\sqrt{2}}\right)^2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] \\ &= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{aligned}$$

4.6

Can

$$|\psi\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$$

be written as a product state?

Ei voida, sillä kyseessä on Bell-tila ($|\Phi^+\rangle$), jota ei voida hajoittaa tekijöihin.

4.7

Find $X \otimes Y |\psi\rangle$, where

$$|\psi\rangle = \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}$$

$$\begin{aligned}
X \otimes Y|\psi\rangle &= (X \otimes Y)\left(\frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}(X \otimes Y)|0\rangle|1\rangle - \frac{1}{\sqrt{2}}(X \otimes Y)|1\rangle|0\rangle \\
&= \frac{1}{\sqrt{2}}(X|0\rangle)(Y|1\rangle) - \frac{1}{\sqrt{2}}(X|1\rangle)(Y|0\rangle) \\
&= \frac{1}{\sqrt{2}}|1\rangle(-i|0\rangle) - \frac{1}{\sqrt{2}}|0\rangle i|1\rangle \\
&= \frac{-i}{\sqrt{2}}(|1\rangle|0\rangle + |0\rangle|1\rangle)
\end{aligned}$$

4.8

Show that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

Ilman yleisyyden menettämistä (wlog) olkoon A ja B matriisit $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Pidetään mielessä, että $(xy)^* = x^*y^*$:

$$\begin{aligned}
(A \otimes B)^\dagger &= \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right)^\dagger \\
&= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}^\dagger
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} (a_{11}b_{11})^* & (a_{11}b_{21})^* & (a_{21}b_{11})^* & (a_{21}b_{21})^* \\ (a_{11}b_{12})^* & (a_{11}b_{22})^* & (a_{21}b_{22})^* & (a_{21}b_{22})^* \\ (a_{12}b_{11})^* & (a_{12}b_{21})^* & (a_{22}b_{11})^* & (a_{22}b_{21})^* \\ (a_{12}b_{12})^* & (a_{12}b_{22})^* & (a_{22}b_{12})^* & (a_{22}b_{22})^* \end{pmatrix} \\
&= \begin{pmatrix} a_{11}^*b_{11}^* & a_{11}^*b_{21}^* & a_{21^*}b_{11}^* & a_{21}^*b_{21}^* \\ a_{11}^*b_{12}^* & a_{11}^*b_{22}^* & a_{21}^*b_{22}^* & a_{21}^*b_{22}^* \\ a_{12}^*b_{11}^* & a_{12}^*b_{21}^* & a_{22}^*b_{11}^* & a_{22}^*b_{21}^* \\ a_{12}^*b_{12}^* & a_{12}^*b_{22}^* & a_{22}^*b_{12}^* & a_{22}^*b_{22}^* \end{pmatrix} \\
&= \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} \otimes \begin{pmatrix} b_{11}^* & b_{21}^* \\ b_{12}^* & b_{22}^* \end{pmatrix} \\
&= A^\dagger \otimes B^\dagger
\end{aligned}
$$

4.9

If

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

find $I \otimes Y |\psi\rangle$.

$$
\begin{aligned}
I \otimes Y |\psi\rangle &= I \otimes Y \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{\sqrt{2}} \left(|0\rangle \left(Y |0\rangle \right) + |1\rangle \left(Y |1\rangle \right) \right) \\
&= \frac{i}{\sqrt{2}} (|01\rangle - |10\rangle)
\end{aligned}
$$

4.10

Calculate the matrix representation of $X \otimes Y$.

$$
\begin{aligned}
X \otimes Y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} && (1) \\
&= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} && (2)
\end{aligned}
$$