Exercise 4 :

X4.1. A family has two children. What is the conditional probability that both are boys, given that at least one of them is a boy?

Answer: The sample space $S = \{(b,b), (b,g), (g,b), (g,g)\}$ and all outcomes considered equally likely.
((b, g) - denotes first child is boy and the second is a girl)

let E : the event that both are boys.
F : the event that at least one of them is a boy.

The probability in question is $P(E|F)$.

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$$= \frac{P(\{(b,b)\})}{P(\{(b,b)(b,g),(g,b)\})}$$

$$= \frac{1/4}{3/4} = \frac{1}{3}. \qquad \square$$

2. Each of three guests in a party throws his/her hat into the centre of a room. The hats are mixed up and each randomly picks up a hat. What is the probability that none of the guests gets his/her own hat?

Solution: Of the 6 permutations $\{(1,2,3)\ (1,3,2)\ (2,1,3)\ (2,3,1),\ (3,2,1)\ (3,1,2)\}$ denoting the hat combinations the guests can get, only the permutations $(2,3,1)$ and $(3,1,2)$ pertain to the event that no one gets his/her own hat.

So, the probability of the event is $\frac{2}{6} = \frac{1}{3}$ $\square$

Note : We have taken these problems and solutions from the book <u>ROSS S.M.</u> Introduction to Probability Models

4.3. In a multiple choice test, a student either knows or guesses the answer in answering a question. p is the probability that the student knows the answer and $(1-p)$ is the probability that he/she guesses. Assuming that ~~the~~ a student who guesses at the answer will get the correct answer with probability $1/m$ where m is the number of multiple choice alternatives.

What is the conditional probability that a student knew the answer to a question given that he/she answered it correctly.

Solution:

let C: the event that the student answers the question correctly

K: the event that he/she actually knows the answer.

$$P(K|C) = \frac{P(KC)}{P(C)}$$

$$= \frac{P(C|K)\,P(K)}{P(C|K)P(K) + P(C|K^c)\,P(K^c)}$$

$$= \frac{1.\,p}{1.\,p + (1/m)\,(1-p)}$$

$$= \frac{m\,p}{1+(m-1)\,p}.$$

So, for example, if $m=5$, $p=\frac{1}{2}$, then the probability that a student knew the answer to a question she/he answered correctly is $5/6$. $\square$.

4.4. Let X be binomially distributed r.v. with parameters n and p.

To calculate mean and variance of X.

Solution: $X \sim Bin(n, p)$

$$p(k) := P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} =$$

$$E[X] = \sum_{k=0}^{n} k\, p(k)$$

$$= \sum_{k=0}^{n} i \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{k\, n!}{(n-k)!\, k!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} \quad [\text{let } i = k-1].$$

$$= np\,[p + (1-p)]^{n-1}$$

$$= np.$$

So, the expected # successes in n independent trials is n times the probability that a trial yields a success. $\square$

Second Solution: Use of indicator function:

let Y_i : i^{th} trial. $\mathbb{1}(Y_i) = \begin{cases} 1 & \text{if } Y_i \text{ is a success} \\ 0 & \text{if } Y_i \text{ is not a success.} \end{cases}$

Then $X = \sum_{i=1}^{n} Y_i$ and so

$$E[X] = E\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} E[Y_i] = n\, E[Y_i].$$

$E[Y_i] = p \cdot 1 + (1-p) \cdot 0 = p$. So $E[X] = np$

$\square$.

Variance of binomial distribution $X \sim B(n, p)$.

Moment generating function $\phi_X(t) := \mathbb{E}[e^{tX}]$

$$\phi_X(t) = \mathbb{E}[e^{tX}]$$
$$= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}$$
$$= (pe^t + 1 - p)^n$$

so $\phi_X'(t) = n(pe^t + 1 - p)^{n-1} pe^t$

so $\mathbb{E}[X] = \phi'(0) = np$

$$\phi''(t) = \begin{cases} n(n-1)(pe^t+1-p)^{n-2}(pe^t)^2 \\ + n(pe^t+1-p)^{n-1} pe^t \end{cases}$$

So $\mathbb{E}[X^2] = \phi''(0) = n(n-1)p^2 + np$

So, the variance of X

$$\mathbb{V}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$
$$= n(n-1)p^2 + np - n^2p^2$$
$$= np(1-p)$$

$$\phi_X(t) = \mathbb{E}[e^{tX}]$$
$$\phi_X'(t) = \frac{d}{dt}\mathbb{E}[e^{tX}]$$
$$= \mathbb{E}\left[\frac{d}{dt}(e^{tX})\right]$$
$$= \mathbb{E}[Xe^{tX}]$$

Hence $\phi_X'(0) = \mathbb{E}[X]$

Similarly

$$\phi''(t) = \frac{d}{dt}\phi'(t)$$
$$= \frac{d}{dt}\mathbb{E}[Xe^{tX}]$$
$$= \mathbb{E}\left[\frac{d}{dt}(X e^{tX})\right]$$
$$= \mathbb{E}[X^2 e^{tX}]$$

So $\phi''(0) = \mathbb{E}[X^2]$

In general, the n^{th} derivative of $\phi(t)$ evaluated at $t=0$ equals $\mathbb{E}[X^n]$

i.e. $\phi^{n}(0) = \mathbb{E}[X^n]$ $n \geq 1$

4.5. X is a Poisson r.v. with parameter λ.

To calculate the mean and variance of X.

Solution: $X \sim P(\lambda)$

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} i \cdot \frac{e^{-\lambda}\lambda^{i}}{i!}$$

$$= \sum_{i=1}^{\infty} \frac{e^{-\lambda}\lambda^{i}}{(i-1)!}$$

$$= \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!}$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} \qquad \left[e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!} \right].$$

$$= \lambda .$$

Variance of Poisson distribution with mean λ.

$$X \sim \mathcal{P}(\lambda).$$

mgf $\phi_t(X) := \mathbb{E}[e^{tX}]$

$$= \sum_{n=0}^{\infty} \frac{e^{tn} e^{-\lambda} \lambda^n}{n!}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= \exp\{\lambda(e^t - 1)\}$$

differentiating,

$$\phi'_X(t) = \lambda e^t \exp\{\lambda(e^t-1)\}$$

$$\phi''_X(t) = (\lambda e^t)^2 \exp\{\lambda(e^t-1)\} + \lambda e^t \exp\{\lambda(e^t-1)\}$$

so $\quad \mathbb{E}[X] = \phi'(0) = \lambda$

$$\mathbb{E}[X^2] = \phi''(0) = \lambda^2 + \lambda$$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \lambda.$$

So both the <u>mean</u> and <u>variance</u> of a Poisson r.v. <u>equal</u> (the parameter of Poisson distribution) $\underline{\lambda}$. $\square$.

4.6 In an urn, there are R red balls and B black balls. What is the probability of drawing j red balls in a total of n draws in each of the following two Cases:

(a) after each draw, we replace the drawn ball, mix the balls up, and draw at random again

(b) after each draw the drawn ball is not replaced.

Solution

(a) The ~~p~~ number of balls remains constant, The probability of drawing j red balls in n draws is given by the binomial distribution $B(n, \frac{R}{R+B})$

$$P(j) = \binom{n}{j}\left(\frac{R}{R+B}\right)^{j}\left(\frac{B}{R+B}\right)^{n-j}$$

(This describes sampling with replacement)

(b). If we draw n times without replacing any ball drawn, then (using a Counting argument)

the probability of $j \leq R$ red balls is

$$P(j) = \frac{\binom{R}{j}\binom{B}{n-j}}{\binom{R+B}{n}}.$$

This is hypergeometric distribution and the underlying model describes Sampling without replacement

So
Sampling without replacement involves Hypergeometric distribution

whereas
Sampling with replacement involves Binomial distribution

$\square$

4.7 The number of typographic errors in a page of a book is Poisson distributed with parameter $\lambda = 1$.

To Calculate

(a) The error probability of at least one error in a page

(b) There are no errors in a page.

Solution

a) X - r.v. denoting the # errors in a page.

$$X \sim P(1)$$

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - e^{-1} \approx 0.633$$

b) $$\mathbb{P}(X = 0) = e^{-1} \approx 0.367$$

$\square$

Mathematical Modelling for Computer Networks
Spring 2013

Exercise 4: Due on 12th April 2013.

Write your answers to the questions briefly and clearly. Please bring a printout (or a handwritten copy) of your answers to the class. You may refer to the book Introduction to Probability by Grinsted and Snell (`http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html`)

1. A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy?

2. Each of three guests in a party throws his/her hat into the centre of a room. The hats are mixed up and each randomly picks up a hat. What is the probability that none of the three guests winds up with his/her own hat?

3. In a multiple choice test a student either knows or guesses the answer in answering a question. Let p be the probability that the student knows the answer and 1-p is the probability that he/she guesses. Assuming that a student who guesses at the answer will get the correct answer with probability 1/m where m is the number of multiple choice alternatives. What is the conditional probability that a student knew the answer to a question given that he/she answered it correctly?

4. Let X be a binomial random variable with parameters n and p. Calculate the mean and variance of X using the mean and variance formula for a random variable.

5. Let X be a Poisson random variable with parameter λ. Calculate the mean and variance of X using the mean and variance formula for a random variable.

6. Urn models are widely used in probabilistic modelling. This question is an example. In an urn there are R red balls and B black balls. What is the probability of drawing j red balls in a total of n draws in each of the following two cases: (a) after each draw we replace the drawn ball, mix the balls up and draw at random again and (b) after each draw the drawn ball is not replaced.

7. The number of typographical errors in a page of a book is Poisson distributed with parameter $\lambda = 1$. Calculate the probability of the events (a) there is at least one error in a page and (b) there are no errors in a page.