Mathematical Modelling for Computer Networks
Spring 2013

Exercise 5: Due on 19th April 2013.

Write your answers to the questions briefly and clearly. Please bring a printout (or a handwritten copy) of your answers to the class. You may refer to the book Introduction to Probability by Grinstead and Snell (`http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html`)

1. Markov inequality, Chebechev inequality and Jensen inequality are basic inequalities in probability. State and derive them.

2. Show that binomial distribution $B(n, p)$ can be approximated by Poisson distribution $P(\lambda)$ when n is large and p is small such that $\lambda = np$

3. Information can be regarded as 'dual' of probability; the information associated with probability p is given by $I(p) = -log\, p$ taken with a suitable base for the logarithm such as $e, 10, 2$. When the base is chosen as 2, the amount of information is given in bits. Given a probability vector $\mathbf{p} = (p_1, \ldots, p_n)$, the entropy of associated with $\mathbf{p}$, denoted by $H(\mathbf{p}) = \mathbf{E}(-log\,(\mathbf{p})) \doteq -\sum_1^n p_i\, log\, p_i$ where $\mathbf{E}$ is the expectation operator wrt the probability vector $\mathbf{p}$. Show that for an arbitrary probability n-vector $\mathbf{p} = (p_1, p_2, ...p_n)$, the entropy $H(\mathbf{p})$ is maximum when all the component probabilities p_i are equal. Interpret the result.

4. Show that Poisson arrivals implies that interarrival times are exponentially distributed.

5. Show that for Poisson arrivals, the number of arrivals in disjoint time intervals are independent.

6. Show that in the merging of two independent Poisson arrival streams with parameters λ_1 and λ_2, the resulting stream is also Poisson with parameter $\lambda_1 + \lambda_2$.

7. The invariant distribution of (an irreducible) Markov chain is a probability vector π such that $\pi = \pi\mathbf{P}$ where $\mathbf{P}$ is one-step transition matrix of the Markov chain. Calculate π for the 2-state Markov chain

$$\mathbf{P} = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

   Evaluate π_0 for the case $\alpha = 0.7$ and $\beta = 0.7$