Exercises V

Labelled algorithms, figures, and chapters refer to the course book.

V-1 (**CLRS 5.1-3**$\star$) Suppose that you want to output 0 with probability 1/2 and 1 with probability 1/2. At your disposal is a procedure BIASED-RANDOM, that outputs either 0 or 1. It outputs 1 with some probability p and 0 with probability $1-p$, where $0 < p < 1$, but you do not know what p is. Give an algorithm that uses BIASED-RANDOM as a subroutine, and returns 0 with probability 1/2 and 1 with probability 1/2. What is the expected running time of your algorithm as a function of p.

V-2 (**CLRS 5.2-5**) Let $A[1..n]$ be an array of n distinct numbers. If $i < j$ and $A[i] > A[j]$, then the pair (i, j) is called an *inversion* of A. Suppose that the elements of A form a uniform random permutation of $(1, 2, \ldots, n)$. Use indicator random variables to compute the expected number of inversions.

V-3 (**CLRS 5.3-3**) Suppose that instead of swapping element $A[i]$ with a random element from the subarray $A[i..n]$, we swapped it with a random element from anywhere in the array:

```
PERMUTE-WITH-ALL(A, n)
for i = 1 to n
      swap A[i] with A[RANDOM(1, n)]
```

Does this code produce a uniform random permutation? Why or why not?

V-4 (**CLRS 5.3-5**$\star$) Prove that in the array P in procedure PERMUTE-BY-SORTING, the probability that all elements are unique is at least $1 - 1/n$.

V-5 (**CLRS 8-4 Water jugs**) Suppose that you are given n red and n blue water jugs, all of different shapes and sizes. All red jugs hold different amounts of water, as do the blue ones. Moreover, for every red jug, there is a blue jug that holds the same amount of water, and vice versa.

It is your task to find a grouping of the jugs into pairs of red and blue jugs that hold the same amount of water. To do so, you may perform the following operation: pick a pair of jugs in which one is red and one is blue, fill the red jug with water, and then pour the water into the blue jug. This operation will tell you whether the red or the blue jug can hold more water, or if they are of the same volume. Assume that such a comparison takes one time unit. Your goal is to find an algorithm that makes a minimum number of comparisons to determine the grouping. Remember that you may not directly compare two red jugs or two blue jugs.

a. Describe a deterministic algorithm that uses $O(n^2)$ comparisons to group the jugs into pairs.

c. Give a randomized algorithm whose expected number of comparisons is $O(n \log n)$, and prove that this bound is correct. What is the worst-case number of comparisons for your algorithm?