There are exactly five biplanes with $k = 11$

Petteri Kaski
Helsinki Institute for Information Technology HIIT
Department of Computer Science
University of Helsinki
P.O. Box 68, 00014 University of Helsinki, Finland
E-mail: `petteri.kaski@cs.helsinki.fi`

Patric R. J. Östergård*
Department of Electrical and Communications Engineering
Helsinki University of Technology
P.O. Box 3000, 02015 TKK, Finland
E-mail: `patric.ostergard@tkk.fi`

Abstract

A biplane is a 2-$(k(k-1)/2+1, k, 2)$ symmetric design. Only sixteen nontrivial biplanes are known: there are exactly nine biplanes with $k < 11$, at least five biplanes with $k = 11$, and at least two biplanes with $k = 13$. It is here shown by exhaustive computer search that the list of five known biplanes with $k = 11$ is complete. This result further implies that there exists no 3-$(57, 12, 2)$ design, no 112_{11} symmetric configuration, and no $(324, 57, 0, 12)$ strongly regular graph. The five biplanes have 16 residual designs, which by the Hall–Connor theorem constitute a complete classification of the 2-$(45, 9, 2)$ designs.

Keywords: biplane; classification; strongly regular graph; symmetric configuration; symmetric design

---

*Supported in part by the Academy of Finland, Grants No. 107493 and 110196.

1 Introduction

A 2-(v, k, λ) *design* is a pair $(V, \mathcal{B})$, where V is a v-element set of *points* and $\mathcal{B}$ is a collection of k-element subsets of V—called *blocks*—such that each 2-element subset of V occurs as a subset in exactly λ blocks. Denoting the number of blocks by b and the number of blocks in which a point occurs by r, direct counting arguments give the following necessary existence conditions:

$$vr = bk, \quad \lambda(v-1) = r(k-1).$$

Designs with $b = v$, called *symmetric* or *square* designs, have been extensively studied [24, 25].

Symmetric designs with $\lambda = 1$ and $\lambda = 2$ are called *projective planes* and *biplanes* [6], respectively. While the projective planes constitute an infinite family of designs, it has been conjectured that only finitely many symmetric designs exist for any fixed $\lambda > 1$.

Currently sixteen nontrivial biplanes are known: there are exactly nine biplanes with $k < 11$, at least five biplanes with $k = 11$, and at least two biplanes with $k = 13$. Despite the results for $k < 11$ dating back three decades, a complete classification of the biplanes with $k = 11$ has resisted attack until the present work. Several studies consider classification of biplanes with $k = 11$ and admitting specific groups of automorphisms [8, 9, 10, 11, 12, 15, 22, 29, 30].

A computer search, described in Section 2, shows that the list of five known biplanes with $k = 11$ is exhaustive.

Theorem 1 *There are exactly five biplanes with* $k = 11$.

The main properties of the $k = 11$ biplanes are reviewed in Section 3. In particular, Theorem 1 has the following two corollaries.

Corollary 2 *There exists no* 3-$(57, 12, 2)$ *design, no* 112_{11} *symmetric configuration, and no* $(324, 57, 0, 12)$ *strongly regular graph.*

By the Hall–Connor theorem [19], the 2-$(45, 9, 2)$ designs are residuals of the biplanes with $k = 11$.

Corollary 3 *There are exactly* 16 2-$(45, 9, 2)$ *designs.*

2 The search

For the standard graph- and design-theoretic terminology, the reader is referred to [4, 5]. Algorithms for classification of designs are discussed in [27].

2.1 Structure of biplanes

It is convenient to first recall some facts about the structure of symmetric designs and biplanes.

A symmetric design has the property that the design obtained by interchanging the roles of points and blocks—called the *dual* of the original design—is also a design with the same parameters. A symmetric design that is isomorphic to its dual is said to be *self-dual*. The parameter $k-\lambda$ is called the *order* of a symmetric design.

In terms of the parameter k, the other parameters v, b, r, λ of a biplane are

$$v = b = \binom{k}{2} + 1, \quad r = k, \quad \lambda = 2.$$

In particular, for $k = 11$ we have $v = b = 56$, $r = 11$, and $\lambda = 2$.

The following observation dates back to work by Hussain [23] and Martinetti [31]. Consider any block B of a biplane. For any two distinct points in B, there is exactly one other block that intersects B in exactly these points. Consequently, the blocks in $\mathcal{B} \setminus \{B\}$ can be indexed by the pairs of distinct points in B. In other words, viewing the points in B as the vertices of a complete graph K_k, the blocks in $\mathcal{B} \setminus \{B\}$ are in a one-to-one correspondence with the edges of K_k.

The structural importance of this correspondence becomes evident when one looks at the points in $V \setminus B$. Namely, each point $p \in V \setminus B$ is incident with $r = k$ blocks, whose corresponding k edges in K_k define a 2-factor of K_k by virtue of each point in B occurring together with p in $\lambda = 2$ blocks. Furthermore, because any two distinct blocks intersect in $\lambda = 2$ points and any two distinct points in $V \setminus B$ occur together in $\lambda = 2$ blocks, it follows that a biplane corresponds to a set of $v - k$ 2-factors of K_k such that

(1) each pair of adjacent edges in K_k occurs in exactly one 2-factor;

(2) each pair of nonadjacent edges in K_k occurs in exactly two 2-factors;

(3) any two 2-factors have exactly two edges in common.

2.2 Structure of the search and seeds

Biplanes can now be classified by considering all sets of 2-factors with the aforementioned properties, one 2-factor at a time, using backtrack search with isomorph rejection. This approach is essentially the one used by Salwach and Mezzaroba to settle the previous case, $k = 9$, thirty years ago [36, 38].

As far as details are concerned, several new ideas are needed to make the search feasible for $k = 11$. Up to isomorphism, there are six 2-factors of K_{11}, characterized by the lengths of their constituent cycles (see Fig. 1):

$$4+4+3, \quad 5+3+3, \quad 4+7, \quad 6+5, \quad 8+3, \quad 11.$$

We call these Type I to Type VI, in this order.

INSERT FIGURE 1 ABOUT HERE

To consider all sets of 2-factors meeting properties (1)–(3), it suffices to split the search into six parts, where in Part P we assume that the set contains at least one 2-factor of Type P and all other 2-factors are of Type P to Type VI. For example, in Part VI this means that all 2-factors are of the same type, Type VI. To enable exact reconstruction of the subsequent intermediate results, the initial labeled 2-factors that we consider are displayed below.

I: {1,2},{1,3},{2,4},{3,4},{5,6},{5,7},{6,7},{8,9},{8,10},{9,11},{10,11};
II: {1,2},{1,3},{2,3},{4,5},{4,6},{5,6},{7,8},{7,9},{8,10},{9,11},{10,11};
III: {1,2},{1,3},{2,4},{3,4},{5,6},{5,7},{6,8},{7,9},{8,10},{9,11},{10,11};
IV: {1,2},{1,3},{2,4},{3,5},{4,5},{6,7},{6,8},{7,9},{8,10},{9,11},{10,11};
V: {1,2},{1,3},{2,3},{4,5},{4,6},{5,7},{6,8},{7,9},{8,10},{9,11},{10,11};
VI: {1,2},{1,3},{2,4},{3,5},{4,6},{5,7},{6,8},{7,9},{8,10},{9,11},{10,11}.

After fixing the first 2-factor, the search space for the remaining 2-factors has not much symmetry left. The orders of the automorphism groups of the six 2-factors are 768, 720, 112, 120, 96, and 22, respectively.

Given the first 2-factor, properties (1)–(3) give us many degrees of freedom for selecting the second 2-factor. The following heuristic selection strategy is aimed at minimizing the size of the whole search space. In each of the six cases, we proceed by selecting the pair of edges indicated with dashed lines in Fig. 1, and consider in turn each 2-factor that contains this pair and does not violate (1)–(3). After the second 2-factor has been added in this manner, there are 114, 137, 319, 426, 569, and 3993 nonisomorphic intermediate solutions in the respective cases. For exact reconstruction, the six selected edge pairs are listed below; we consider the 2-factors containing the selected pair in lexicographic order, whereby a set of 2-factors is rejected if it is isomorphic to one considered earlier.

$$\begin{array}{rlrlrl} \mathrm{I:} & \{1,2\},\{3,4\}; & \mathrm{II:} & \{7,8\},\{9,11\}; & \mathrm{III:} & \{1,2\},\{3,4\}; \\ \mathrm{IV:} & \{6,7\},\{10,11\}; & \mathrm{V:} & \{4,11\},\{7,8\}; & \mathrm{VI:} & \{1,6\},\{1,7\}. \end{array}$$

We utilize a similar heuristic when selecting the third 2-factor. Namely, we first generate all 2-factors extending the two fixed 2-factors, after which we select the (lexicographic minimum) edge pair occurring in the least nonzero number of candidate 2-factors and having one edge in the intersection of the two fixed 2-factors. When selecting the third 2-factor, we consider (in lexicographic order) all the candidate 2-factors containing the selected edge pair.

After the third 2-factor has been added, there are 360371, 370839, 1213721, 1299483, 1645428, and 9215229 intermediate solutions, respectively. The first five counts are after isomorph rejection; no isomorph rejection was carried out in the last case. These sets of three 2-factors are the seeds for the main search.

2.3 Main search

From the perspective of biplanes, a seed corresponds to a completion of three points in $V \setminus B$ incident with a common block. The main part of the search continues from here and attempts to complete the points of two full blocks in addition to B.

The setup is illustrated by the partial point-block incidence matrix in Fig. 2. The top part of the matrix consists of the block B and an associated incidence matrix of K_{11}, the middle part contains the three seed 2-factors, and the bottom part illustrates the part that the main search proceeds to complete; the two blocks to be completed are indicated by down-arrows in the top row.

INSERT FIGURE 2 ABOUT HERE

Given a seed, we can choose which blocks—that is, edges of K_{11}—to complete. For reasons of efficiency, we require that one of the two edges must occur in all three 2-factors of the seed; by construction each seed has a unique such edge. Furthermore, we require that the two edges have a vertex in common.

Let us look at the structural implications of the aforementioned two requirements. Let $\{x, y\}$ and $\{x, z\}$ be the selected edges, where x is the common vertex. Observe that the 2-factors of K_{11} containing $\{x, y\}$ or $\{x, z\}$ (or both) are partitioned into 17 classes based on the pair of edges incident with x. To complete the blocks corresponding to $\{x, y\}$ and $\{x, z\}$, we must choose exactly one 2-factor from each class. The seed fixes the choices for 3 classes, leaving 14 classes for subsequent search. We choose for completion a pair $\{x, y\}, \{x, z\}$ such that the total number of 2-factors in the remaining classes is minimized.

We carry out the main search as a basic backtrack search that selects exactly one 2-factor from each class using a minimum branching heuristic. More precisely, when choosing the next 2-factor, we find a class with the minimum number of 2-factors, and then consider each 2-factor in turn from this class. Whenever a 2-factor is chosen, we filter out those 2-factors from the other classes that do not have exactly two edges in common with the chosen 2-factor. For performance reasons, this is the only type of filtering we carry out during the main search. Initially the classes of 2-factors are constructed so that they do not violate properties (1)–(3), however.

One small modification to the described scheme turned out to improve the overall performance: instead of considering all 14 classes, we disregard the initially largest class from consideration, also when selecting the two blocks (edges) to be completed.

If a solution is found in the main search, then a final search is employed that attempts to complete the remaining $56 - 11 - 3 - 13 = 29$ points. The amount of time needed for this final completion part of the search is negligible compared with the main search. The classification is complete after carrying out isomorph rejection among the scarce biplanes found.

2.4 Search statistics

The number of biplanes constructed from the seeds in each of the six parts of the search is 169, 14, 0, 0, 0, and 0, respectively. After isomorph rejection, only the five known biplanes remain.

An extensive amount of computing time was required to carry out the main search, which was executed in parallel on two networks of Linux PCs using the batch system `autoson` [32]. A total of 316 different computers were at some point running the search. The total amount of computer time consumed by each of the six parts was 1.9, 1.9, 8.5, 3.3, 2.0, and 5.5 years, respectively. The reported times are the sums of running times of individual batch jobs as recorded by the operating system. Due to the fact that a number of different CPU architectures were employed to execute the search, these numbers should only be viewed as an indication of the order of magnitude of computational resources required.

In terms of real time, the main search was carried out in a little over two months. The initial seed classification and final completion parts of the search require a few hours of computer time with unoptimized code.

2.5 Consistency checks

Correctness is a fundamental concern in an extensive computer search such as the present one. In particular, the possibility of hardware errors cannot be ruled out. (For example, during the course of this search, one malfunctioning computer was discovered and subsequently decommissioned.) To gain a degree of confidence in the correctness of the search, we have performed the following consistency checks.

To ensure that the search is exhaustive, it is imperative that the seeds have been correctly classified. Therefore both authors independently implemented the seed classification software and carried out the classification with identical results.

In checking the subsequent search, we employ a scheme that counts in two different ways the biplanes constructed in each of the six parts. The first count is simply the total number of biplanes constructed. The second count—described in detail in the appendix—relies on the classified seeds and biplanes. The two counts agree in the parts of the search in which biplanes are discovered; that is, Parts I and II.

3 The biplanes with $k = 11$

We here survey the main properties of the five biplanes with $k = 11$ in the order they were discovered (the years of publication are indicated). The biplanes are named $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5$, and are all self-dual. A detailed account of the first four of these appears in [37]. The 2-factor distributions of the biplanes are presented in Table 1.

Many authors (late 1960s) ($\mathcal{B}_1$): This biplane was first discovered as a design by Hall, Lane, and Wales [20] and independently by Rudvalis (unpublished). However, its existence also follows directly from the existence of a strongly regular graph with parameters $(56, 10, 0, 2)$, discovered in several studies in the late 1960s. Namely, if A is the adjacency matrix of a $(v, k, 0, 2)$ strongly regular graph, then $A+I$ is an incidence matrix for a biplane with v points and block size $k+1$. Such a graph was found independently by Gewirtz [16, 17], who also proved that it is unique; and Sims (see [16, pp. 925–926]). In fact, this biplane is also very closely linked to several group-theoretical results obtained in the 1960s, including [14, 21, 34]. The automorphism group of this design has order 80640. Its incidence matrix has 3-rank (that is, rank over the finite field $\mathbb{F}_3$) 20.

Salwach and Mezzaroba (1979) [37] ($\mathcal{B}_2$): The automorphism group of this design has order 288. Its incidence matrix has 3-rank 22.

Denniston (1980) [13] ($\mathcal{B}_3, \mathcal{B}_4$): The automorphism groups of these designs have order 144 ($\mathcal{B}_3$) and 64 ($\mathcal{B}_4$). Their incidence matrices have 3-rank 26 and 24, respectively.

Janko and Tran van Trung (1986) [26] ($\mathcal{B}_5$): The automorphism group of this design has order 24. Its incidence matrix has 3-rank 26.

INSERT TABLE 1 ABOUT HERE

This classification of biplanes makes it possible to address several other related problems. We conclude this paper by addressing a few existence problems that are now settled.

Cameron [7] proved that if a symmetric 2-(v, k, λ) design is extendible—that is, is a derived design of a 3-$(v+1, k+1, \lambda)$ design—then one of the following holds: (a) $v = 4\lambda + 3$, $k = 2\lambda + 1$ (Hadamard design); (b) $v = (\lambda + 2)(\lambda^2 + 4\lambda + 2)$, $k = \lambda^2 + 3\lambda + 1$; or (c) $v = 495$, $k = 39$, $\lambda = 3$. When $\lambda = 2$ in Case (b), we have biplanes with $k = 11$. Key and Tonchev [28] studied the known biplanes and proved that none of these can be extended; this had been showed for one of these biplanes already by Hall (see [7]). Note that the proof published in [3] is in error. The current study completes this work and proves that there exists no 3-$(57, 12, 2)$ design. Via a result due to Mesner [33], this also implies the nonexistence of a strongly regular graph with parameters $(324, 57, 0, 12)$.

One early motivation for studying biplanes with these parameters was the fact that a projective plane of order 10—that is, a symmetric 2-$(111, 11, 1)$ design—could have been constructed from a certain biplane of order 9; see [2, 35]. We now know that there is no projective plane of order 10. However, we can obtain another closely related (nonexistence) result from the current classification.

If we relax the definition of a 2-$(v, k, 1)$ symmetric design and require that each 2-element subset of points occur in *at most* one block, then we get the definition of a v_k *symmetric configuration*. A projective plane of order 10 is a 111_{11} symmetric configuration. It has not been known whether a 112_{11} symmetric configuration exists. Gropp [18, Lemma 3.7] proved that if a 112_{11} symmetric configuration exists, then it can be used to construct a biplane of order 9 that is not isomorphic to one of the five known biplanes. Consequently, the current work proves that there is no 112_{11} symmetric configuration.

The Hall–Connor theorem [19] implies that any 2-$(k(k-3)/2+1, k-2, 2)$ design is a residual design of a biplane. Thus, by forming all the residuals of the biplanes with $k = 11$ and rejecting isomorphs, we obtain a complete classification of the 2-$(45, 9, 2)$ designs. There are 16 isomorphism classes of such designs.

As regards continuation of the present classification work, the Bruck–Ryser–Chowla theorem shows that there exists no biplane with $k = 12$, so the next open case is $k = 13$, with two known (dual) biplanes [1]. Unless new ideas are developed for attacking that case, its resolution is probably not to be seen in the foreseeable future. A more accessible intermediate goal is to classify the biplanes with $k = 13$ admitting a nontrivial automorphism group.

Acknowledgments

The authors thank Arunas Rudvalis for comments on the early history of biplanes with $k = 11$, Chester Salwach for sending a copy of [38], and Jani Jaakkola and Pekka Niklander at the Department of Computer Science of University of Helsinki and Juhani Markula and Jukka Seppänen at the Computing Centre of Helsinki University of Technology for providing computing resources.

Appendix: A consistency check

This appendix describes a counting technique that we use to check that the correct number of biplanes are constructed in the search. To this end, we first require an appropriate representation for the isomorphism classes of seeds and biplanes as orbits of a group action.

Let $\{0,1\}^v$ be the set of all ordered v-tuples of 0s and 1s. A biplane is represented as a v-element subset of $\{0,1\}^v$ consisting of the rows of a point-block incidence matrix. A seed is represented as a $(k+3)$-element subset of $\{0,1\}^v$ consisting of the points incident with one block and three additional points incident with a common block. (For example, consider the first 14 rows in Fig. 2.)

For a v-tuple $x \in \{0,1\}^v$, denote by $x(i)$ the ith entry of x, $i = 1, 2, \ldots, v$. Denote by $\mathrm{Sym}(\{1,2,\ldots,v\})$ the symmetric group on $\{1,2,\ldots,v\}$. Let an element $g \in \mathrm{Sym}(\{1,2,\ldots,v\})$ act on x by permuting the entries so that gx is the v-tuple defined for all $i = 1, 2, \ldots, v$ by $(gx)(i) = x(g^{-1}(i))$. Extend the action to an elementwise action on sets consisting of v-tuples in $\{0,1\}^v$. Note that the orbits of this action on representations of biplanes (seeds) correspond to the usual isomorphism classes. Extend the action elementwise also to ordered pairs (X, S), where $X, S \subseteq \{0,1\}^v$. Let $\mathrm{Aut}(X) = \{g \in \mathrm{Sym}(\{1,2,\ldots,v\}) : gX = X\}$ and $\mathrm{Aut}(X,S) = \{g \in \mathrm{Sym}(\{1,2,\ldots,v\}) : gX = X \text{ and } gS = S\}$.

Based on the classified seeds and biplanes, we now count the total number of biplanes that should be constructed in Part P of the search. In precise terms, taking the sum over each seed $S \subseteq \{0,1\}^v$ in Part P, we count the number of ordered pairs (X, S) such that (a) $X \subseteq \{0,1\}^v$ is a biplane; (b) $S \subseteq X$; and (c) the 2-factor types in $X \setminus S$ are admissible in Part P. By the orbit-stabilizer theorem, the $\mathrm{Aut}(S)$-orbit of each such pair (X, S) consists of $|\mathrm{Aut}(S)|/|\mathrm{Aut}(X,S)|$ pairs. Thus, for every orbit of such pairs, it suffices to accumulate a counter by $|\mathrm{Aut}(S)|/|\mathrm{Aut}(X,S)|$ 'units'.

We carry out the count as follows. For each classified biplane, X', we find all pairs (X', S') such that (a) $S' \subseteq X'$; (b) S' is isomorphic to a seed; and (c) the 2-factor types in $X' \setminus S'$ are admissible. For each such pair, we accumulate the counter by $|\mathrm{Aut}(S')|/|\mathrm{Aut}(X')|$ units.

To see that the correct count is reached, consider an arbitrary pair (X, S) such that (a) X is a biplane; (b) $S \subseteq X$ is a seed; and (c) the 2-factor types in $X \setminus S$ are admissible. Clearly, X is isomorphic to exactly one classified biplane X'. By the orbit-stabilizer theorem, we find exactly $|\mathrm{Aut}(X')|/|\mathrm{Aut}(X', S')|$

pairs (X', S') in the orbit of (X, S). Because each pair accumulates the count by $|\mathrm{Aut}(S')|/|\mathrm{Aut}(X')|$ units, the total accumulation for the orbit of (X, S) is precisely the required $|\mathrm{Aut}(S)|/|\mathrm{Aut}(X, S)|$ units.

As regards implementation in practice, a biplane with $k = 11$ contains $v(v-1)\binom{k-2}{3} = 258720$ candidate seeds—that is—substructures induced by the points incident with a block and three additional points incident with a common block. For each classified biplane, we consider each candidate seed in turn. We accumulate the count if the candidate seed is isomorphic to a seed and the 2-factor types are admissible. In this way we find that 169 biplanes should be constructed in Part I of the search, and that 14 biplanes should be constructed in Part II. These are precisely the numbers of biplanes constructed in the search.

References

[1] M. Aschbacher, On collineation groups of symmetric block designs, J Combin Theory Ser A 11 (1971), 272–281.

[2] E. F. Assmus, Jr., J. A. Mezzaroba, and C. J. Salwach, "Planes and biplanes," Higher combinatorics, M. Aigner (Editor), D. Reidel Publishing Company, Dordrecht, 1977, pp. 205–212.

[3] B. Bagchi, No extendable biplane of order 9, J Combin Theory Ser A 49 (1988), 1–12; and 57 (1991), 162.

[4] T. Beth, D. Jungnickel, and H. Lenz, Design theory, 2 vols., Cambridge University Press, Cambridge, 2nd ed., 1999.

[5] B. Bollobás, Modern graph theory, Springer, New York, 1998.

[6] P. J. Cameron, Biplanes, Math Z 131 (1973), 85–101.

[7] P. J. Cameron, Extending symmetric designs, J Combin Theory Ser A 14 (1973), 215–220.

[8] V. Ćepulić, On biplanes $(56, 11, 2)$ with automorphism groups of order four, Glas Mat Ser III 31(51) (1996), 201–207.

[9] V. Ćepulić and M. Essert, Biplanes $(56, 11, 2)$ with automorphism group $Z_2 \times Z_2$ fixing some point, J Combin Theory Ser A 48 (1988), 239–246.

[10] V. Ćepulić and M. Essert, Biplanes $(56, 11, 2)$ with automorphisms of order 4 fixing some point, Discrete Math 71 (1988), 9–17.

[11] V. Ćepulić and M. Essert, Biplanes and their automorphisms, Studia Sci Math Hungar 24 (1989), 437–446.

[12] V. Ćepulić and M. Essert, Biplanes $(56, 11, 2)$ with involutory automorphism fixing 14 points, Glas Mat Ser III 31(51) (1996), 25–38.

[13] R. H. F. Denniston, On biplanes with 56 points, Ars Combin 9 (1980), 161–179.

[14] W. L. Edge, Some implications of the geometry of the 21-point plane, Math Z 87 (1965), 348–362.

[15] M. Essert and L. Marangunić, Biplanes $(56, 11, 2)$ with a fixed-point-free involutory automorphism, Glas Mat Ser III 40(60) (2005), 1–11.

[16] A. Gewirtz, Graphs with maximal even girth, Canad J Math 21 (1969), 915–934.

[17] A. Gewirtz, The uniqueness of $g(2, 2, 10, 56)$, Trans New York Acad Sci (2) 31 (1969), 656–675.

[18] H. Gropp, On the existence and non-existence of configurations n_k, J Combin Inform System Sci 15 (1990), 34–48.

[19] M. Hall, Jr., and W. S. Connor, An embedding theorem for balanced incomplete block designs, Canad J Math 6 (1953), 35–41.

[20] M. Hall, Jr., R. Lane, and D. Wales, Designs from permutation groups, J Combin Theory 8 (1970), 12–22.

[21] D. G. Higman and C. C. Sims, A simple group of order 44,352,000, Math Z 105 (1968), 110–113.

[22] K. Horvatić-Baldasar, E. Kramer, and I. Matulić-Bedenić, A contribution to the classification of biplanes of order 9, Rad Jugoslav Akad Znan Umjet, No 428 (1987), 25–29.

[23] Q. M. Hussain, On the totality of the solutions for the symmetrical incomplete block designs: $\lambda = 2$, $k = 5$ or 6, Sankhyā 7 (1945), 204–208.

[24] Y. J. Ionin and M. S. Shrikhande, Combinatorics of symmetric designs, Cambridge University Press, Cambridge, 2006.

[25] Y. J. Ionin and Tran van Trung, "Symmetric designs," Handbook of combinatorial designs, C. J. Colbourn and J. H. Dinitz (Editors), Chapman & Hall / CRC, Boca Raton, 2nd ed., 2007, pp. 110–124.

[26] Z. Janko and Tran van Trung, A new biplane of order 9 with a small automorphism group, J Combin Theory Ser A 42 (1986), 305–309.

[27] P. Kaski and P. R. J. Östergård, Classification algorithms for codes and designs, Springer, Berlin, 2006.

[28] J. D. Key and V. D. Tonchev, "Computational results for the known biplanes of order 9," Geometry, combinatorial designs and related structures, J. W. P. Hirschfeld, S. S. Magliveras, and M. J. de Resmini (Editors), Cambridge University Press, Cambridge, 1997, pp. 113–122.

[29] L. Marangunić, The classification of biplanes $(56, 11, 2)$ admitting an automorphism of order 8 which fixes some point, Rad Mat 2 (1986), 99–111.

[30] L. Marangunić, Biplanes $(56, 11, 2)$ with involutory collineation fixing 6 points, J Combin Theory Ser A 54 (1990), 149–163.

[31] V. Martinetti, Un' osservazione relativa alla configurazione di Kummer, Giornale di Matematiche di Battaglini 34 (1896), 192–194.

[32] B. D. McKay, `autoson`—a distributed batch system for UNIX workstation networks (version 1.3), Technical Report TR-CS-96-03, Computer Science Department, Australian National University, Canberra, 1996.

[33] D. M. Mesner, Negative latin square designs, Institute of Statistics Mimeo Series No 410, Department of Statistics, North Carolina State University, 1964.

[34] S. Montague, On rank 3 groups with a multiply transitive constituent, J Algebra 14 (1970), 506–522.

[35] C. J. Salwach, Planes, biplanes, and their codes, Amer Math Monthly 88 (1981), 106–125.

[36] C. J. Salwach and J. A. Mezzaroba, The four biplanes with $k = 9$, J Combin Theory Ser A 24 (1978), 141–145.

[37] C. J. Salwach and J. A. Mezzaroba, The four known biplanes with $k = 11$, Internat J Math Math Sci 2 (1979), 251–260.

[38] C. J. Salwach and J. A. Mezzaroba, An exhaustive search technique for combinatorial problems, Bell Laboratories internal technical memorandum, 1983.

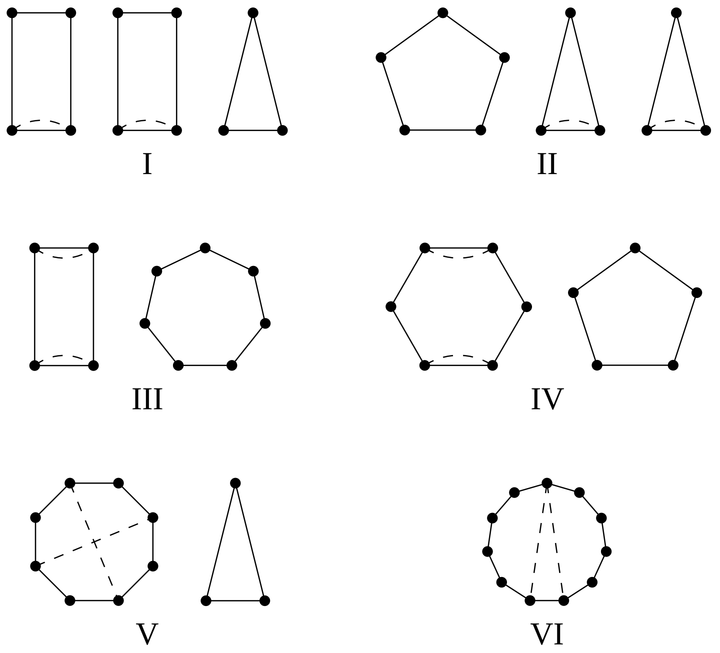

Figure 1: The six 2-factors of K_{11}

$$\begin{array}{c|ll}
B & \downarrow\qquad\qquad\quad\downarrow & \\
1 & 1111111111000000000000000000000000000000000000000000000 & \\
1 & 1000000000111111111000000000000000000000000000000000000 & \\
1 & 0100000000100000000111111110000000000000000000000000000 & \\
1 & 0010000000010000000100000001111111000000000000000000000 & \\
1 & 0001000000001000000010000001000000111111000000000000000 & \\
1 & 0000100000000100000001000000100000100000111110000000000 & K_{11} \\
1 & 0000010000000010000000100000010000010000100001111000000 & \\
1 & 0000001000000001000000010000001000001000010001000111000 & \\
1 & 0000000100000000100000001000000100000100001000100100110 & \\
1 & 0000000010000000010000000100000010000010000100010010101 & \\
1 & 0000000001000000001000000010000001000001000010001001011 & \\
\hline
0 & 1100000000010000000100000000000000110000100000000110011 & \\
0 & 1010000000100000000100000000000000001100010010011000100 & \text{Seed} \\
0 & 1001000000000100000000001010001010010000001000010001000 & \\
\hline
0 & 1000001000001000000001000010010100000010000010100010000 & \\
0 & 1000000001000010000000011001100000001000000100100000001 & \\
0 & 1000010000000000100010000100011000000001010100000000010 & \\
0 & 1000100000000000001001100001000010000001000001000100100 & \\
0 & 1000000100000001000000100100100001000110100000000001000 & \\
0 & 1000000010000000010010010000000101100000001001001000000 & \\
0 & 0100000001001000010001000000101000000100000001010000010 & \text{Main} \\
0 & 0000001001000010001110000000001000000010101000000000100 & \\
0 & 0000100001100000010000001000010001010010010000000100000 & \\
0 & 0001000001010001000010100000000100000000000110010100000 & \\
0 & 0000010001100100000000010000000110000101100000000010000 & \\
0 & 0000000101010100000000000110010000101000000001000000100 & \\
0 & 0000000011001000100100100000000010100000010000100001000 & \\
0 & 0010000001000001100001000101000000010000001000001010000 &
\end{array}$$

Figure 2: An illustration of the main search

Table 1: The 2-factor distributions

Biplane	4+4+3	5+3+3	7+4	6+5	8+3	11	#
$\mathcal{B}_1$	45						56
$\mathcal{B}_2$	45						2
	13	8	24				18
	13		8		8	16	36
$\mathcal{B}_3$	36				9		2
	5		12	8	8	12	36
	4		8		17	16	18
$\mathcal{B}_4$	21					24	4
	13	8	24				4
	7	4	10	2	4	18	32
	5	4	8	4	6	18	16
$\mathcal{B}_5$	5	4	8	4	2	22	6
	1	4	8		2	30	6
	1	1	6	4	8	25	24
		4	10	6	7	18	12
		3	6	3	6	27	8