Introduction to Information-Theoretic Modeling

Teemu Roos

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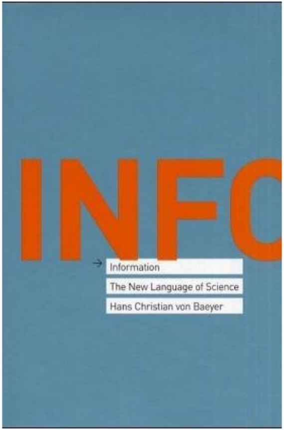

"Whether on the internet, encoded in radio waves or coursing through wires, information is all around us. Our senses record it, our brains process it and our genes pass it on. But what exactly is information? Can it be analysed and measured? [...] a concept that could soon become as central to science as space, time mass or energy."

1 Course Outline

1. Course Outline

2. What is Coding?


INFORMATION


INFORMATION


INFORMATION

1. Course Outline
   - Course details
   - What is Information?
   - Why Information?
   - Information vs. Complexity
   - Information Theory
2. What is Coding?
3. Symbol Codes
4. Entropy and Information
5. Kolmogorov complexity

Introduction to Information-Theoretic Modeling

- A short course, 2×3 h.

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- Lecture notes: `www.cs.helsinki.fi/teemu.roos/brazil.pdf`

Resources

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- Etymology: *informare* = give form, 14th century.

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- *knowledge [...], intelligence, news, facts, data, [...], (as nucleotides in DNA or binary digits in a computer program) [...], a signal [...], a numerical quantity that measures the uncertainty in the outcome of an experiment to be performed.* (source: Merriam-Webster).

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- This course: measuring *the amount* of information in data, and using such measures for automatically buiding *models*.

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- Need to *store, transmit, and process* information efficiently.
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- Computer science: make things automatic (intelligent).

Information vs. Complexity

Is complexity the same as information?

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Is there a lot of *information* in a random string? **No.**

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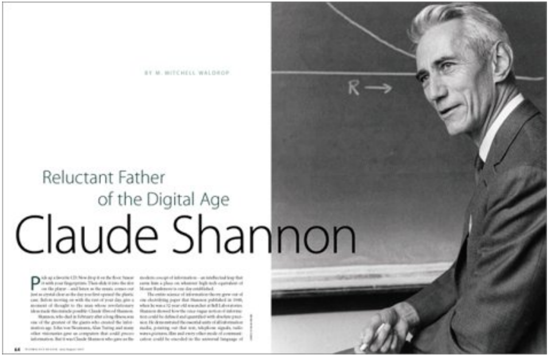
$$\begin{aligned} \textit{Complexity} &= \textit{Information} + \textit{Noise} \\ &= \textit{Regularity} + \textit{Randomness} \\ &= \textit{Algorithm} + \textit{Compressed file} \end{aligned}$$

Information Theory



"The real birth of modern information theory can be traced to the publication in 1948 of Claude Shannon's *"The Mathematical Theory of Communication"* in the Bell System Technical Journal. "
(Encyclopædia Britannica)

Course Topics

Information Theory:

- entropy and information, bits,

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Modeling:

- statistical models,
- complexity (in data and models),
- over-fitting, Occam's Razor, and MDL Principle.

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2. What is Coding?
   - Dots and Dashes
   - Codes as Mappings
   - Data Compression
3. Symbol Codes
4. Entropy and Information
5. Kolmogorov complexity

Coding Game

Form groups of 3–4 persons. Each group constructs a *code* for the letters A–Z by using as *code-words* unique sequences of dots • and dashes (—) like "•", "— •", " — • — —", etc.

A	_______	G	_______	M	_______	S	_______	Y	_______
B	_______	H	_______	N	_______	T	_______	Z	_______
C	_______	I	_______	O	_______	U	_______		
D	_______	J	_______	P	_______	V	_______		
E	_______	K	_______	Q	_______	W	_______		
F	_______	L	_______	R	_______	X	_______		

Coding Game

Use your code to *encode* the message
"`WHAT DOES THIS HAVE TO DO WITH INFORMATION`".

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• • • — — — • • •: $1+1+1+2+2+2+1+1+1=12$.

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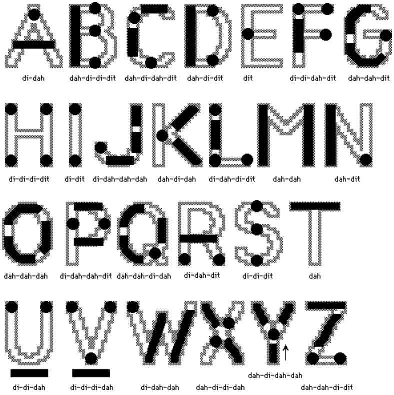
$\bullet\bullet\bullet$ — — — $\bullet\bullet\bullet$: $1+1+1+2+2+2+1+1+1=12$.

The *coding rate* of your code is the length of the encoded message divided by the length of the original message, including spaces (42).

Coding Game



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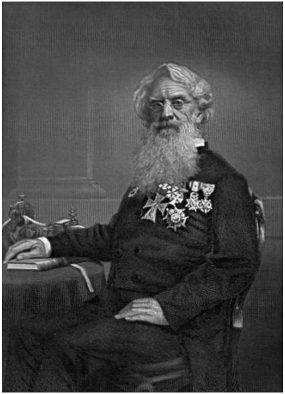

Samuel F.M. Morse (1791–1872)

Coding Game

`WHAT DOES THIS HAVE TO DO WITH INFORMATION`

Coding Game

```
WHAT DOES THIS HAVE TO DO WITH INFORMATION

.-- .... .- -   -.. --- . ...   - .... .. ...
.... .- ...- .   - ---   -.. ---   .-- .. - ....
.. -. ..-. --- .-. -- .- - .. --- -.
```

Coding Game

```
WHAT DOES THIS HAVE TO DO WITH INFORMATION
```

```
.-- .... .- -   -.. --- . ...   - .... .. ...
.... .- ...- .   - ---   -.. ---   .-- .. - ....
.. -. ..-. --- .-. -- .- - .. --- -.
```

51 dots, 36 dashes, 7 spaces: $51 + 72 + 14 = 137$ units.

Coding Game

```
WHAT DOES THIS HAVE TO DO WITH INFORMATION
```

```
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.. -. ..-. --- .-. -- .- - .. --- -.
```

51 dots, 36 dashes, 7 spaces: $51 + 72 + 14 = 137$ units.

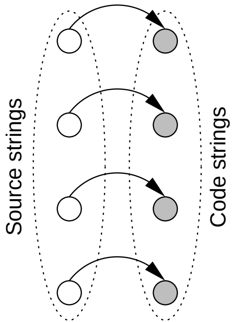
Morse code

Coding rate: $\dfrac{137}{42} \approx 3.26$

Did you do better or worse? Why?

Codes as Mappings

Lossless compression:
injective mapping



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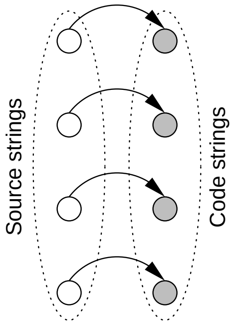

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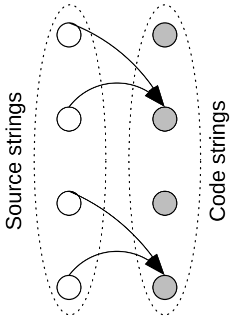

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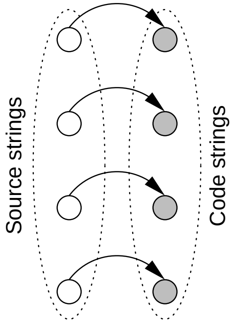

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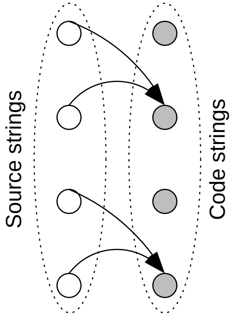

Only *lossless* codes are *uniquely decodable*.

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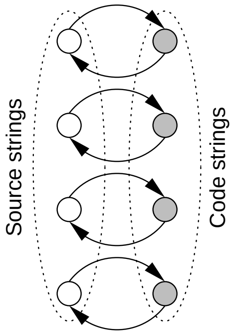

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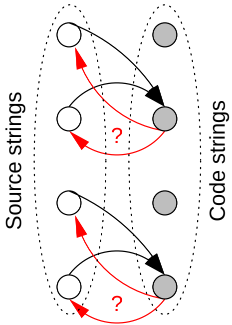

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Examples

general purpose	**`gzip`**
	`bzip`

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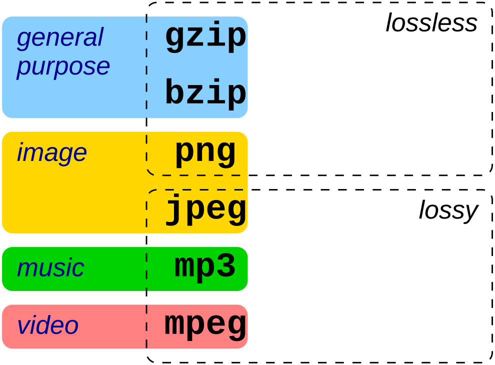
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Examples

		compression ratio	
general purpose	**gzip**	~ 1 : 3	*lossless*
	bzip	~ 1 : 3.5	*lossless*
image	**png**	~ 1 : 2.5	*lossless*
	jpeg	~ 1 : 25	*lossy*
music	**mp3**	~ 1 : 12	*lossy*
video	**mpeg**	~ 1 : 30	*lossy*

Compression

Is it always possible to compress data?

Theorem

The proportion of binary strings compressible by more than k bits is less than 2^{-k}.

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$\square$

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Less than 50 % of files are compressible by more than one bit.

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Less than 1 % of files are compressible by more than 7 bits.

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$$\frac{2^{n-k} - 1}{2^n} < \frac{2^{n-k}}{2^n} = 2^{-k}.$$

□

Less than 0.0000000000000000000000000001 % of files are compressible by 100 bits.

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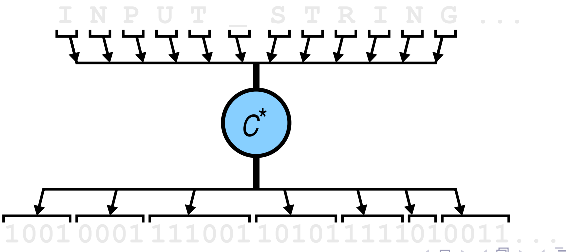
The **extension** of code C is the mapping $C^* : \mathcal{X}^* \to \{0,1\}^*$ obtained by concatenating the codewords $C(x_i)$ for each source symbol x_i:

$$C^*(x_1, x_2, \ldots, x_n) = C(x_1)C(x_2)\ldots C(x_n) \ .$$



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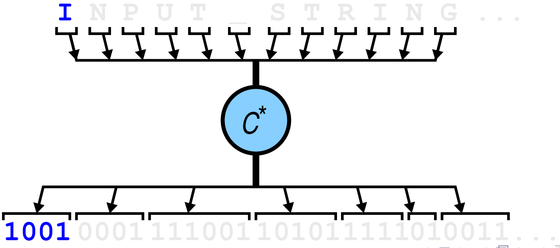

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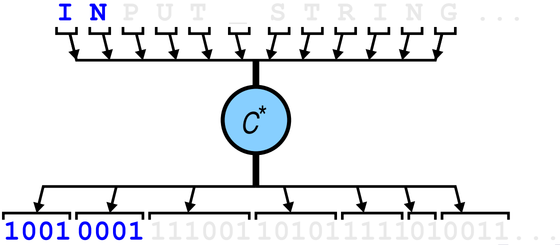

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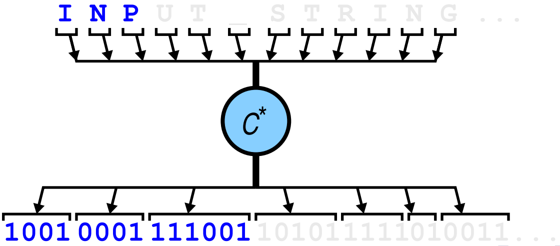

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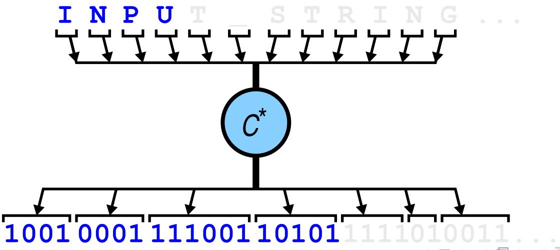

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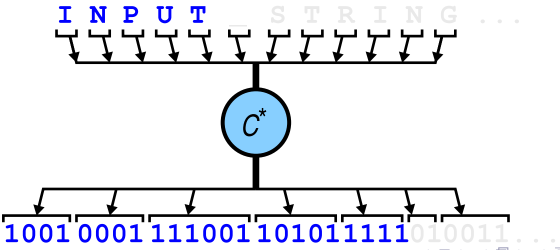

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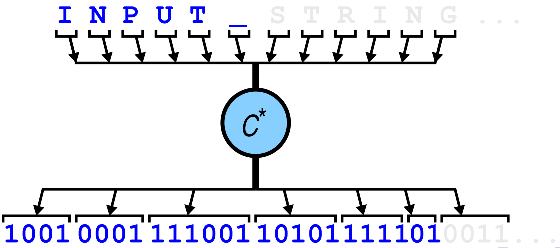

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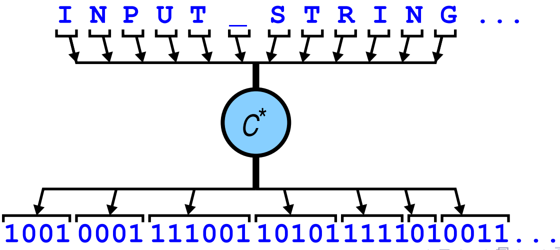

Decodable Codes

Decodable Code

Code C is (uniquely) **decodable** iff its extension C^* is a one-to-one mapping, i.e., iff

$$(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) \ \Rightarrow \ C^*(x_1, \ldots, x_n) \neq C^*(y_1, \ldots, y_n) \ .$$

Decodable Codes

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Code C is (uniquely) **decodable** iff its extension C^* is a one-to-one mapping, i.e., iff

$$(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) \;\Rightarrow\; C^*(x_1, \ldots, x_n) \neq C^*(y_1, \ldots, y_n) \; .$$

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- X A code with codewords $\{0, 1, 10, 11\}$ is *not* uniquely decodable: What does 10 mean?
- √ A code with codewords $\{00, 01, 10, 11\}$ *is* uniquely decodable: Each pair of bits can be decoded individually.
- √ A code with codewords $\{0, 01, 011, 0111\}$ is also uniquely decodable: What does 0011 mean?

Prefix Codes

An important subset of decodable codes is the set of **prefix(-free) codes**.

Prefix Code

A code $C : \mathcal{X} \to \{0, 1\}^*$ is called a **prefix code** iff no codeword is a prefix of another.

It is easily seen that all prefix codes are uniquely decodable: each symbol can be decoded as soon as its codeword is read. Therefore, prefix codes are also called *instantaneous* codes.

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- ✗ A code with codewords $\{0, 01, 011, 0111\}$ is uniquely decodable *but not prefix-free*: e.g., 0 is a prefix of 01.
- √ A code with codewords $\{0, 10, 110, 111\}$ *is* prefix-free.

Kraft Inequality

The codeword lengths of a prefix codes satisfy the following important property.

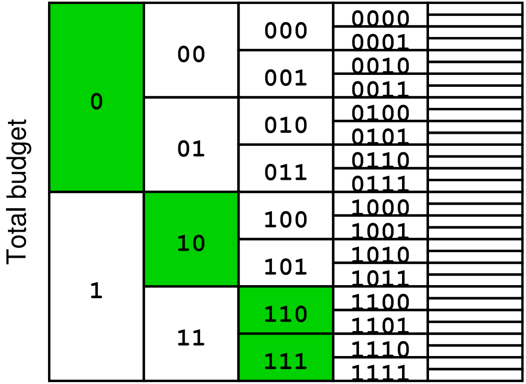
Kraft Inequality

The codeword lengths $\ell_1, \ldots, \ell_m$ of any (binary) prefix code satisfy

$$\sum_{i=1}^{m} 2^{-\ell_i} \leq 1 \quad .$$

Conversely, given a set of codeword lengths that satisfy this inequality, there is a prefix code with these codeword lengths.

Kraft Inequality



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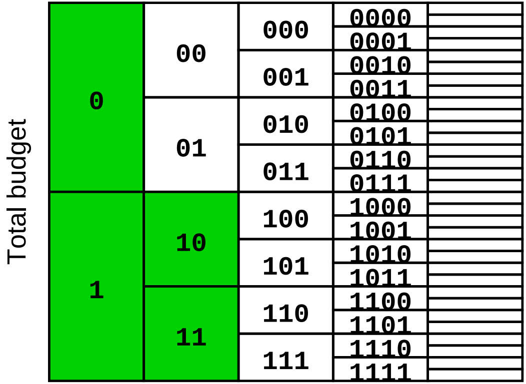
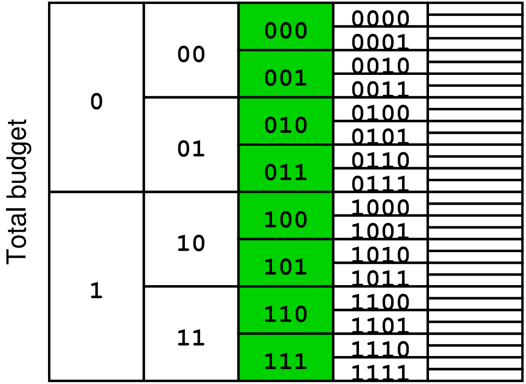


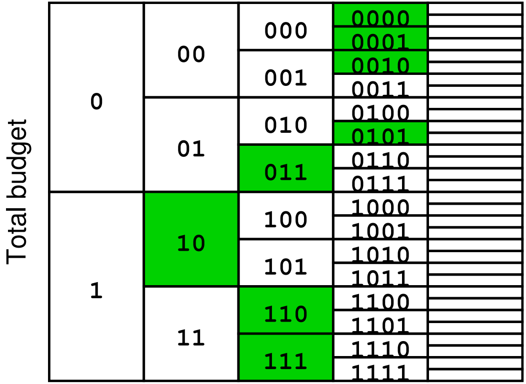
X Kraft inequality violated. $\Rightarrow$ Not decodable.

Kraft Inequality



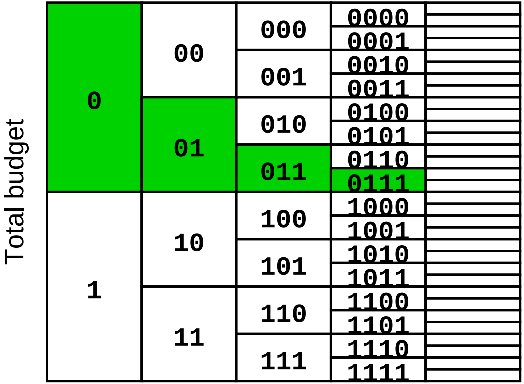
√ Fixed-length code

Kraft Inequality



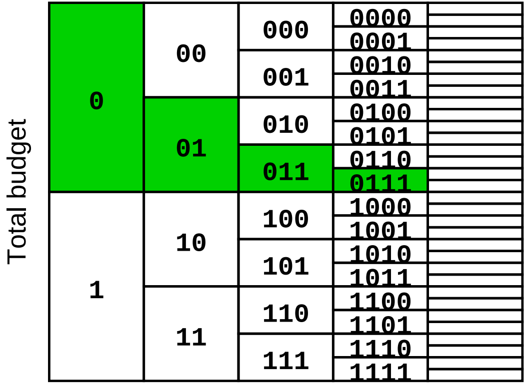
√ Decodable & prefix-free

Kraft Inequality



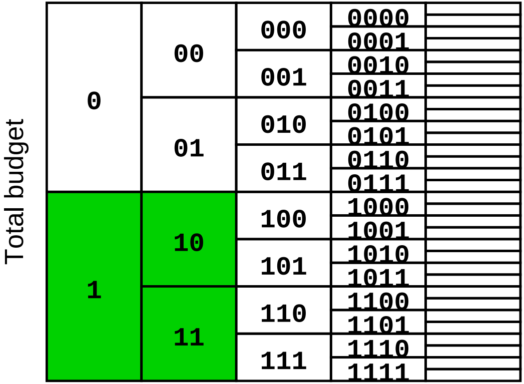
Kraft? Decodable? Prefix-free?

Kraft Inequality



Kraft? √ Decodable? √ Prefix-free? X

Kraft Inequality



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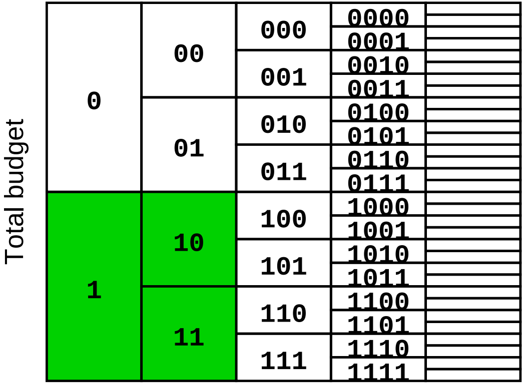

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Question: What if the inequality is satisfied strictly, i.e., the sum of the terms in the sum equals *less* than one:

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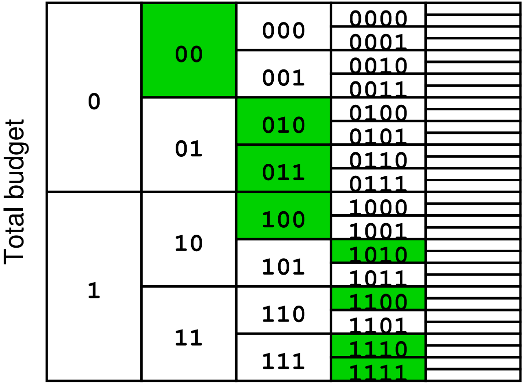
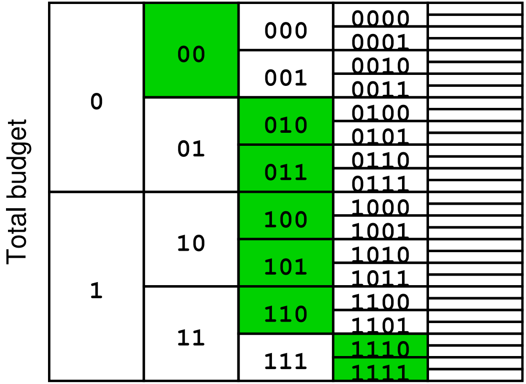
Then it is possible to make the codewords shorter and still have a decodable (prefix) code.

Kraft Inequality



Not all of budget used. $\Rightarrow$ Some codewords can be made shorter.

Kraft Inequality



"Kraft tight" / complete code.

Kraft–McMillan Theorem

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The codeword lengths $\ell_1, \ldots, \ell_m$ of any **uniquely decodable** (binary) code satisfy

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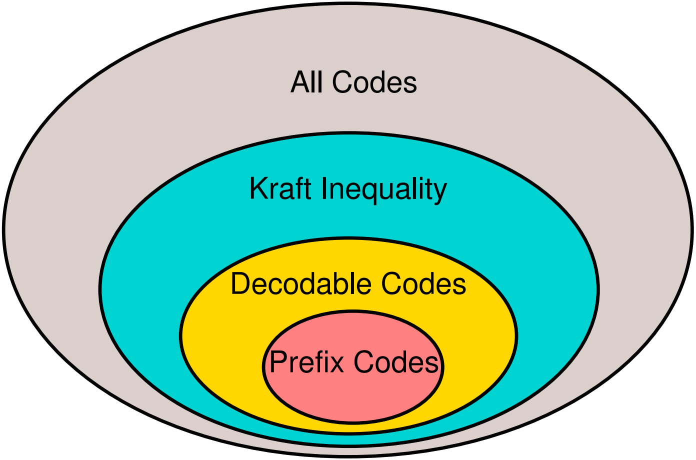
Kraft-McMillan Theorem & Codes



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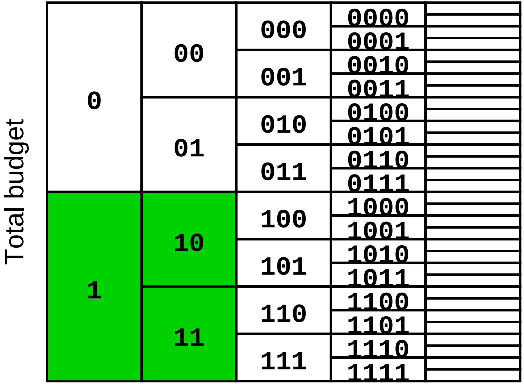

Kraft? √ Decodable? X Prefix-free? X

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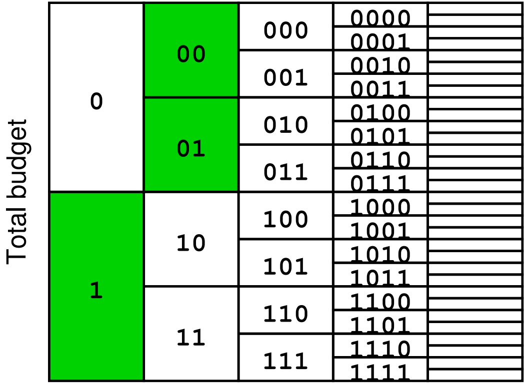

Kraft? √ Decodable? √ Prefix-free? √

Expected Code-length

Now we can tell which codes are decodable, prefix-free, etc.

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Out of two decodable (prefix-free) codes, which one is better?

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For the purpose of **data compression**, the answer is clearly the code that yields the **shortest code-length**.

We consider the expected (per-symbol) code-length:

$$E[\ell(C(X))] = \sum_{x \in \mathcal{X}} p(x)\,\ell(C(x)) \ .$$

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Important Observation

Probability distributions are codes are probability distributions!

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Based on the unification of codes and distributions, we can write

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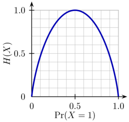
$\Rightarrow$ Information theory (entropy, Kullback-Leibler divergence, ...)


1.0
0.5
0
$H(X)$
0
0.5
1.0
$\Pr(X = 1)$

Entropy

Given a discrete random variable X with pmf p_X, we can measure the amount of "surprise" associated with each outcome $x \in \mathcal{X}$ by the quantity

$$I_X(x) = \log_2 \frac{1}{p_X(x)} \quad .$$

The less likely an outcome is, the more surprised we are to observe it. (The point in the log-scale will become clear shortly.)

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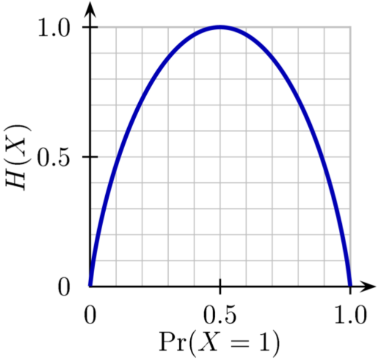
The **entropy** of X measures the *expected* amount of "surprise":

$$H(X) = E[I_X(X)] = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{1}{p_X(x)} \quad .$$

Binary Entropy Function

For binary-valued X, with $p = p_X(1) = 1 - p_X(0)$, we have

$$H(X) = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p} \ .$$



More Entropies

1. the **joint entropy** of two (or more) random variables:

$$H(X,Y) = \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} p_{X,Y}(x,y) \log_2 \frac{1}{p_{X,Y}(x,y)} \ ,$$

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3. and the **conditional entropy**:

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The conditional entropy $H(X \mid Y)$ measures the *expected* uncertainty about X when the value Y is known.

Chain Rule of Entropy

Remember the chain rule of probability:

$$p_{X,Y}(x, y) = p_Y(y) \times p_{X|Y}(x \mid y) \ .$$

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Logarithmic scale makes entropy **additive**.

Mutual Information

The **mutual information**

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measures the average decrease in uncertainty about X when the value of Y becomes known.

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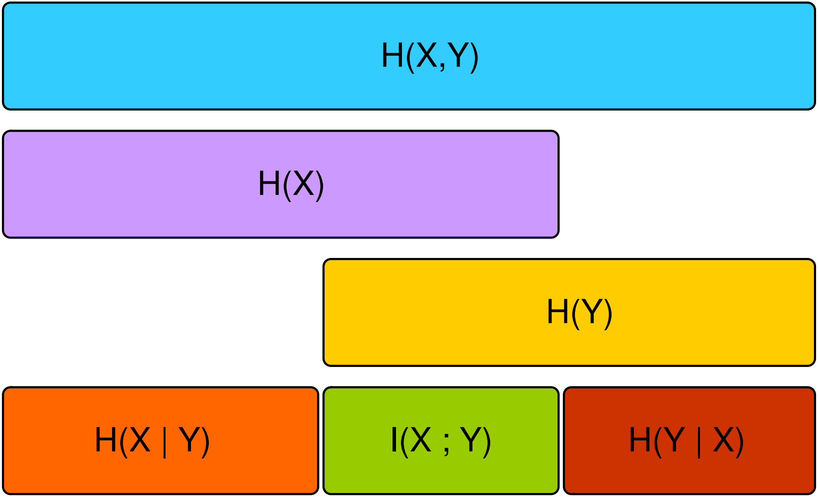
On the average, X gives as much information about Y as Y gives about X.

Relationships between Entropies



Time for a break?

Kullback-Leibler Divergence

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The *relative entropy* or **Kullback-Leibler divergence** between (discrete) distributions p_X and q_X is defined as

$$D(p_X \parallel q_X) = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{p_X(x)}{q_X(x)} \quad .$$

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(We consider $p_X(x) \log_2 \frac{p_X(x)}{q_X(x)} = 0$ whenever $p_X(x) = 0$.)

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Information Inquality

For any two (discrete) distributions p_X and q_X, we have

$$D(p_X \parallel q_X) \geq 0$$

with equality iff $p_X(x) = q_X(x)$ for all $x \in \mathcal{X}$.

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Proof.

$$\begin{aligned} I(X\,;\,Y) &= H(X) - H(X \mid Y) \\ &= H(X) + H(Y) - H(X,Y) \\ &= \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} p_{X,Y}(x,y) \log_2 \frac{p_{X,Y}(x,y)}{p_X(x)\, p_Y(y)} \\ &= D(p_{X,Y} \parallel p_X p_Y) \geq 0 \enspace . \end{aligned}$$

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In addition, $D(p_{X,Y} \parallel p_X p_Y) = 0$ iff $p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$ for all $x \in \mathcal{X}, y \in \mathcal{Y}$. This means that variables X and Y are *independent* iff $I(X\ ;\ Y) = 0$.

Properties of Entropy

Properties of entropy:

1. $H(X) \geq 0$

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   Proof. $p_X(x) \leq 1 \Rightarrow \log_2 \frac{1}{p_X(x)} \geq 0.$

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   Proof. Let $u_X(x) = \frac{1}{|\mathcal{X}|}$ be the uniform distribution over $\mathcal{X}$.

$$0 \leq D(p_X \parallel u_X) = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{p_X(x)}{u_X(x)} = \log_2 |\mathcal{X}| - H(X) \quad .$$

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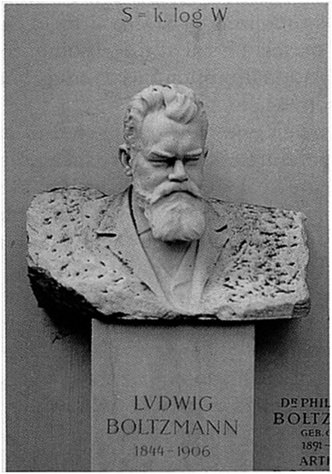
   A **combinatorial** approach to the definition of information (Boltzmann, 1896; Hartley, 1928; Kolmogorov, 1965):

$$S = k \ln W \quad .$$

Ludvig Boltzmann (1844–1906)



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   Proof. $p_X(x) \leq 1 \Rightarrow \log_2 \frac{1}{p_X(x)} \geq 0.$

2. $H(X) \leq \log_2 |\mathcal{X}|$

   A **combinatorial** approach to the definition of information (Boltzmann, 1896; Hartley, 1928; Kolmogorov, 1965):

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   On the average, knowing another r.v. can only reduce uncertainty about X. However, note that $H(X \mid Y = y)$ may be greater than $H(X)$ for some y — "contradicting evidence".

Entropy Lower Bound

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$$E[\ell(X)] \geq H(X) \quad .$$

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Proof.

$$\begin{aligned}
E[\ell(X)] &= \sum_{x \in \mathcal{X}} p(x)\,\ell(x) \\
&= \sum_{x \in \mathcal{X}} p(x) \log_2 \frac{1}{q(x)} \qquad \boxed{q(x) = 2^{-\ell(x)}} \\
&= \sum_{x \in \mathcal{X}} p(x) \left[\log_2 \frac{p(x)}{q(x)} + \log_2 \frac{1}{p(x)}\right] \\
&= D(p \parallel q) + H(X) \geq 0 \quad .
\end{aligned}$$

Entropy Lower Bound

So what have we learned?

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So what have we learned? For ("Kraft-tight") decodable symbols codes:

1. $E[\ell(X)] = H(X) + D(p \parallel q)$, where $q(x) = 2^{-\ell(x)}$.
2. $E[\ell(X)] \geq H(X)$.
3. If $\ell(x) = \log_2 \frac{1}{p(x)}$, then $E[\ell(X)] = H(X)$. **Optimal!**

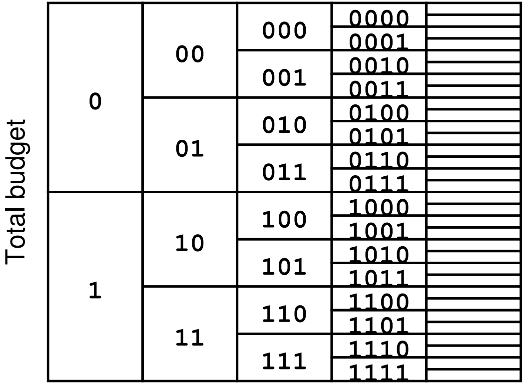
Exercise 5.

5. Let the source distribution p be given by the table below. What are the optimal codeword lengths under p? Can you construct the actual codewords so that the code is prefix-free?

Ex. 5

x	A	B	C	D	E
$p(x)$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$

Exercise 5.


Total budget
0
1
00
01
10
11
000
001
010
011
100
101
110
111
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111

Codelengths and Probabilities

A problem with the $\ell(x) = \log_2 \frac{1}{p(x)}$ codeword choice is the requirement that codeword lengths must be **integers** (try to think about a codeword with length 0.123, for instance).

Codelengths and Probabilities

A problem with the $\ell(x) = \log_2 \frac{1}{p(x)}$ codeword choice is the requirement that codeword lengths must be **integers** (try to think about a codeword with length 0.123, for instance).

The simplest solution is to round upwards:

Shannon's Code

Given a pmf, the **Shannon code** has the codeword lengths

$$\ell(x) = \left\lceil \log_2 \frac{1}{p(x)} \right\rceil \quad \text{for all } x \in \mathcal{X}.$$

Alice in Wonderland

Shannon's code: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$
a	0.0644	3.9	4
b	0.0108	6.5	7
c	0.0178	5.8	6
d	0.0359	4.7	5
e	0.0991	3.3	4
f	0.0147	6.0	7
g	0.0184	5.7	6
h	0.0535	4.2	5
i	0.0551	4.1	5
j	0.0011	9.8	10
k	0.0083	6.8	7
l	0.0343	4.8	5
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y	0.0165	5.9	6
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$$H(X) = 4.03$$

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1. Sort by probability.

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a	0.0644	3.9	4
o	0.0598	4.0	5
i	0.0551	4.1	5
h	0.0535	4.2	5
n	0.0516	4.2	5
s	0.0475	4.3	5
r	0.0401	4.6	5
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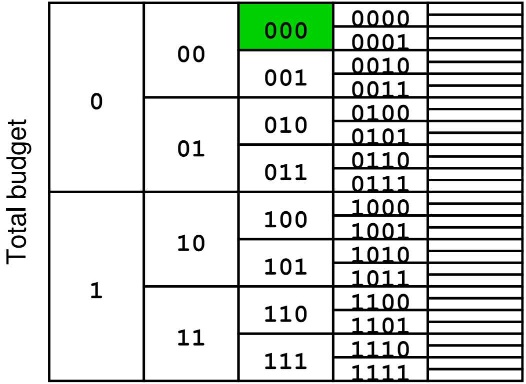
Shannon (1948):

1. Sort by probability.
2. Choose codewords in order, avoiding prefixes. ("Kraft table"!)

Shannon's code: Example



Codeword lengths $(3, 4, 4, 4, 5, 5, 5, 5, \ldots, 10, 10, 11)$

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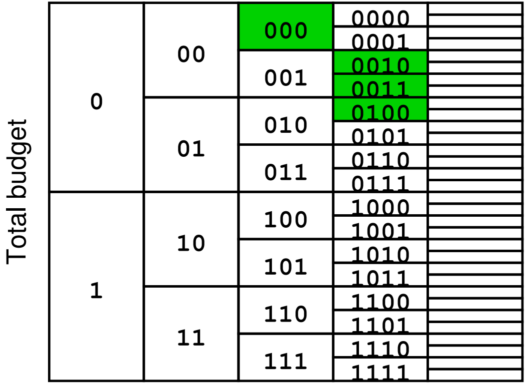

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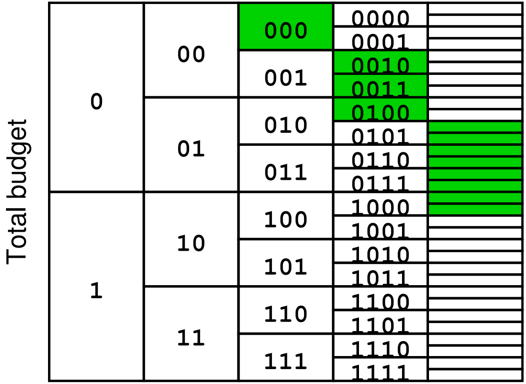

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	0.2111	2.2	3	000
e	0.0991	3.3	4	0010
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$$H(X) = 4.03$$

$$E[\ell(X)] = 4.60$$

$$E[\ell(X)] - H(X) = 0.57$$

Shannon's code

The expected codeword length of Shannon's code is

$$
\begin{aligned}
E\left[\ell(X)\right] &= E\left[\left\lceil \log_2 \frac{1}{p(X)} \right\rceil\right] \\
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Is this optimal? Not necessarily — Huffman!

Kolmogorov complexity

Is the string

$$1010101010101010101\ldots10$$

'simple' or 'complex'?

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(One) answer: Simple because it can be described easily:

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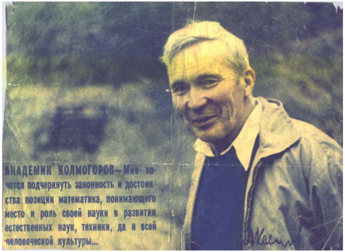
Remark: We should be careful in how we define *describing*; for instance, "to compute by an algorithm" (a formal procedure that eventually halts).

Kolmogorov complexity



Kolmogorov complexity

A.N. Kolmogorov

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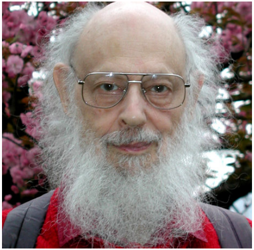

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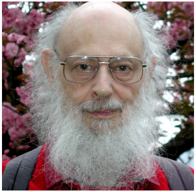
A.N. Kolmogorov



R.J. Solomonoff

Kolmogorov complexity

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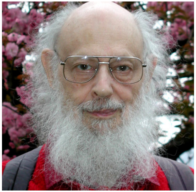

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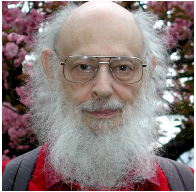

R.J. Solomonoff

G.J. Chaitin

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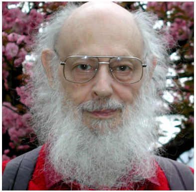

R.J. Solomonoff

G.J. Chaitin

Kolmogorov-Solomonoff-Chaitin complexity

Kolmogorov complexity

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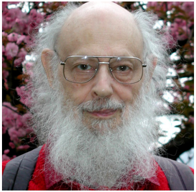

R.J. Solomonoff

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Kolmogorov-Solomonoff-Chaitin complexity

$\longrightarrow$ **Kolmogorov complexity**

Kolmogorov complexity

`echo <x> | gzip - | wc -c` # times 8 (for bits)

source string, x		$\ell(C(x))$	**ratio**
$aaa \dots a$	$(10000 \times a)$	368	27.2 : 1

Kolmogorov complexity

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$aabaabbbbabb\ldots$	(10000 random digits)	13456	0.74 : 1
$abababab\ldots ab$	$(5000 \times ab)$	368	27.2 : 1

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Kolmogorov complexity

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```

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Strings that follow a rule can be compressed?

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$aaabbabbabb\ldots$	(π, 0–4 $\mapsto a$, 5–9 $\mapsto b$)	13416	0.74 : 1

π follows a rule but isn't compressible!

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Perhaps the problem is in `gzip`? It would be possible to write a *specific program* that compresses π.

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But what does it mean to compress *an individual string*???

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An individual string is “simple” (not “complex”) if it can be compressed using a *pre-specified* program.

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Which program? `gzip` isn't good at compressing images (nor digits of π).

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We can use several programs as long as we prefix the file with a code indicating the used program.

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What about new programs?

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Kolmogorov-kompleksisuus: mritelm

Let $U : \{0,1\}^* \to \{0,1\}^* \cup ⌛$ be a computer that given a program $\omega \in \{0,1\}^*$ either prints out a finite output $U(\omega) \in \{0,1\}^*$ or keeps computing forever. In the latter case, we say that the output $U(\omega)$ is undefined (⌛).

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Kolmogorov complexity

Given a string $x \in \{0,1\}^*$, let $\omega^*(x)$ be the *shortest* program such that

$$U(\omega^*(x)) = x \ .$$

The **Kolmogorov complexity** of x is the length of program $\omega^*(x)$:

$$K_U(x) = \min_{p\,:\,U(p)=x} |p| \ .$$

Kolmogorov complexity: basic properties

Let U and V be two computers. If computer U is 'rich' enough it can 'emulate' computer V.

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Universal computer

Computer U is said to be **universal** if for *any* other computer V, there exists a "translation program" $\tau \in \{0,1\}^*$ such that for all programs ω, we have

$$U(\tau\omega) = V(\omega) \quad .$$

Examples

The following are (in principle) universal computers

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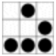

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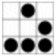

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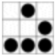
8. Game of Life 
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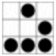

Each of these can emulate any of the others.

In contrast, `gzip` (or rather, `gunzip`) is a **non-universal** computer.

Kolmogorov complexity: basic principles

Lemma: For any *universal computer* U and any other computer V we have

$$K_U(x) \leq K_V(x) + C \quad ,$$

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Proof: Let τ be a translation program that translates the programs of V into programs of U, and let $\omega_V^*(x)$ be the shortest program such that $V(\omega_V^*(x)) = x$. Then, $U(\tau\omega_V^*(x)) = x$, and hence

$$K_U(x) \leq |\tau\omega_V^*(x)| = |\omega_V^*(x)| + |\tau| = K_V(X) + |\tau| \quad . \qquad \square$$

Invariance theorem

From now on, we consider the Kolmogorov complexity, K_U, defined using a *universal computer* U.

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Kolmogorov complexity is (up to an additive constant) invariant wrt. the choice of the universal computer. In other words, for any two universal computers U and V, there is a constant $C > 0$ such that

$$|K_U(x) - K_V(x)| \leq C \quad \text{for any } x \in \{0,1\}^* \ .$$

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Proof: Let $\tau_{V \to U}$ be a program that translates programs of V into programs of U so that $U(\tau\omega) = V(\omega)$ for all ω. Then $K_U(x) \le K_V(x) + |\tau_{V \to U}|$ for all x. Similarly, $K_V(x) \le K_U(x) + |\tau_{U \to V}|$ for all x. The theorem follows by setting $C = \max\{|\tau_{V \to U}|, |\tau_{U \to V}|\}$. $\square$

Conditional Kolmogorov complexity

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The **conditional Kolmogorov complexity** is the length of the shortest program to convert input y into output x:

$$K_U(x \mid y) = \min\{|\omega| \quad : \ U(\bar{y}\,\omega) = x\}$$

where $\bar{y}$ is a "self-delimiting" description of y.

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Uniform upper bounds

The following upper bound holds for all x:

$$K_U(x \mid |x|) \leq |x| + C$$

where C is a constant independent of x.

Examples

Let $n = |x|$.

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   (The estimated entropy of English is about 1.3 bits per symbol.)
4. $K_U(\text{fractal}) = C$.
   Program: `print the number of interations until` $z_{n+1} = z_n^2 + c > T$`.`

Examples

Martin-Löf randomness

Examples (contd.):

5. $K_U(x \mid n) \approx n$ for almost all $x \in \{0,1\}^n$.

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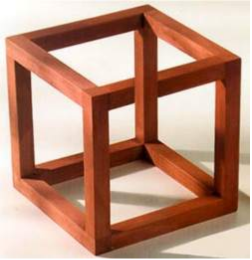
Consequence of point 5: A sequence of coin tosses is Martin-Löf random with high probability.

Berry paradox



What is the least natural number that cannot be described using thirteen words?

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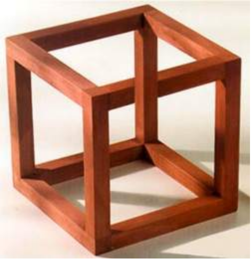

What is the least natural number that cannot be described using thirteen words?

Whatever the number is, we have just described(?) it using thirteen words!

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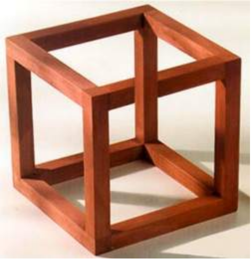

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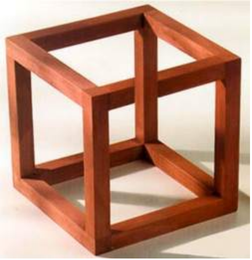

The least *uninteresting* natural number?

Whatever it is, such a number is quite interesting.

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There is no algorithmic way to compute $K_U(x)$.

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Contradiction follows by choosing M greater than the Kolmogorov complexity of the above program. Hence, $K_U(x)$ cannot be computable. □

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- Uncomputable.

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To summarize:

- Kolmogorov complexity, $K_U(x)$, is the length of the shortest program, ω, such that $U(\omega) = x$.
- The choice of the universal computer, U, affects the definition by an additive constant independent of x.
- Uncomputable.
- Enables the definition of randomness of *individual* strings.

Tomorrow

Tomorrow's plan:

1. Occam's Razor,

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Thanks for your attention. Now, let's have a few caipirinhas!