Accuracy of Estimates for Dynamic Properties of Queueing Systems in Interactive Simulation

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***Abstract*— Interactive graphics allows the analyst to view the output sequences during the simulation run. Therefore, the dynamic properties of queueing systems are more and more often examined. In this paper we propose an approach to characterize and estimate dynamic properties so that existing methods for estimating variances of means can be applied. The approach is based on transforming the original output sequence into several new sequences, the means of which together summarize important aspects of the dynamic properties in the original sequence. In addition, we propose a way of assessing the accuracy of estimates that is appropriate in interactive simulation. The method introduced is shown to have a sound theoretical basis. We also report experimental results of applying the method.**

***Keywords*— Interactive Simulation, Dynamic Properties of Queueing Systems, Run Length Control, Confidence Estimation, Simultaneous Inference, Moving Range**

I. Introduction

Interactive graphics that modern workstations provide has created a new epoch in simulation through allowing the analyst to view the output sequence during the simulation run. This possibility has moved, at least partially, the focus from the traditional time-independent characteristics - such as the average, variance, percentiles, or proportions - to time-dependent characteristics of the queueing system simulated. The systems, even if they are in stochastic equilibrium, have also time-dependent or dynamic properties which are of great practical importance. Typically, the dynamic properties are related to the variation of traffic or load during some period of time.

Time series plot is a powerful method for examining the dynamic properties of queueing systems. Many simulation packages can today display the simulation output as a time series during the simulation run. In interactive simulation the decision when the simulation should be terminated is

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left to the analyst. However, the package should provide relevant information so that the analyst can reliably decide when important phenomena have been seen. The current packages do not provide such pieces of information when dynamic properties are examined.

Even in interactive simulation of stochastic systems, the study is only a programming exercise, if statistical techniques are not used to assess the accuracy of estimated indices. The most widely used method for the assessment of accuracy is to estimate a confidence interval. The methods for confidence interval estimation of the steady-state mean are well-known. Available methods are described in many textbooks and survey articles; e.g. [1; 2; 3; 4]. As explained in [4], these methods can be used to estimate a confidence interval for other indices if the indices can be estimated through cumulative statistics.

When dynamic properties are examined, the fundamental problem is the characterization: Which set of indices properly summarizes the dynamic properties? The second problem is, how we can assess the accuracy of estimated indices. In this paper we examine one possible way to characterize and estimate dynamic properties of queueing systems. The variety of dynamic properties that different queueing systems may have is so huge that a uniform approach is impossible. Yet, we believe that the approach we propose can be used in many situations. The approach is based on the following idea: *The original output sequence* $\{X_i\}_{i=1}^{N}$ *is transformed into new sequences* $\{Y_{j,i}\}_{i=1}^{N_j}$; $j = 1, \ldots, m$. *The objective of the transforms is that the means of the new sequences together summarize the important aspects of the dynamic properties in the original sequence.*

Because of the diversity in dynamic properties, we are convinced that several means are simultaneously needed to characterize the dynamic properties. The rationale of translating the problem into simultaneous estimation of several means is that the estimates are based on cumulative statistics. Therefore, we can use the Bonferroni inequality (see Box 1) and the existing methods for estimating the variance of the sample mean.

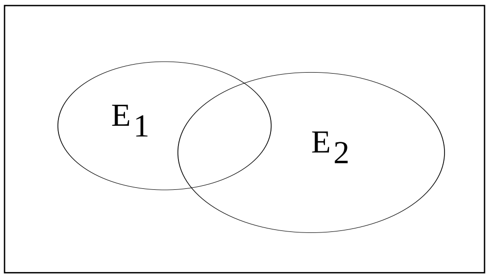
We characterize the dynamic properties in the original output sequence by three indices: the steady-state mean, the average of short-term variation, and the average of medium-term variation. The steady-state mean does not actually characterize the dynamic properties, but it gives a yardstick for the two other averages. The accuracy of estimated indices is assessed through estimating the probability that the estimates have at least the prespecified relative precisions. The methods described in this study are implemented in the prototype of an interactive simulation tool.

In Section II we introduce the transforms and indices which are used to characterize the dynamic properties of the original output sequence. In Section III we describe the method for estimating the

Box 1: Bonferroni Inequality

Venn diagram is an easy way to show that the probability of event E_1 or event E_2 satisfies:

$$\Pr\{E_1 \vee E_2\} \leq \Pr\{E_1\} + \Pr\{E_2\} \ .$$



Venn Diagram

Hence $\Pr\{E_1 \vee \ldots \vee E_M\} \leq \sum_{m=1}^{M} \Pr\{E_m\}$.
Let the parameters of interest be θ_m, $m = 1, \ldots, M$, and their estimated confidence intervals be $[\widehat{L}_m, \widehat{U}_m]$. Let S_M be the statement that the true (unknown) value of θ_m lies within the estimated confidence interval $[\widehat{L}_m, \widehat{U}_m]$. Let E_m denote the event that the statement S_m is incorrect. Suppose that we know the probabilities of events E_m; $\Pr\{E_m\} = \Pr\{\text{statement } S_m \text{ is incorrect}\}$.
The Bonferroni inequality provides us a lower bound for the probability that each statement S_m, $m = 1, \ldots, M$, is (simultaneously) correct:

$$\begin{aligned} \Pr\{S_1, \ldots, \text{ and } S_M \text{ are correct}\} &= 1 - \Pr\{E_1 \vee \ldots \vee E_M\} \\ &\geq 1 - \sum_{m=1}^{M} \Pr\{E_m\} \end{aligned}$$

Mathematical details can be found, for example, in R. G. Miller: *Simultaneous Statistical Inference.* New York: McGraw-Hill, 1966.

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probabilities. In Section IV we show that the method has a sound theoretical basis. In addition, we report empirical results of assessing the accuracy.

II. Characterization of Dynamic Properties

The variety of dynamic properties makes it difficult or even impossible to discuss those properties in general. Therefore, we concentrate on properties that are important but also easy enough to analyze. By easiness we mean that the estimators of the properties are based on cumulative statistics. This allows us to apply the existing methods in assessing the accuracy of estimates. The methods will be introduced in Section III. In the current section we consider the characterization. We start by presenting an example. Then we discuss the theoretical background. Finally we introduce our indices and transforms.

A. An Example: Measured Packet Rate

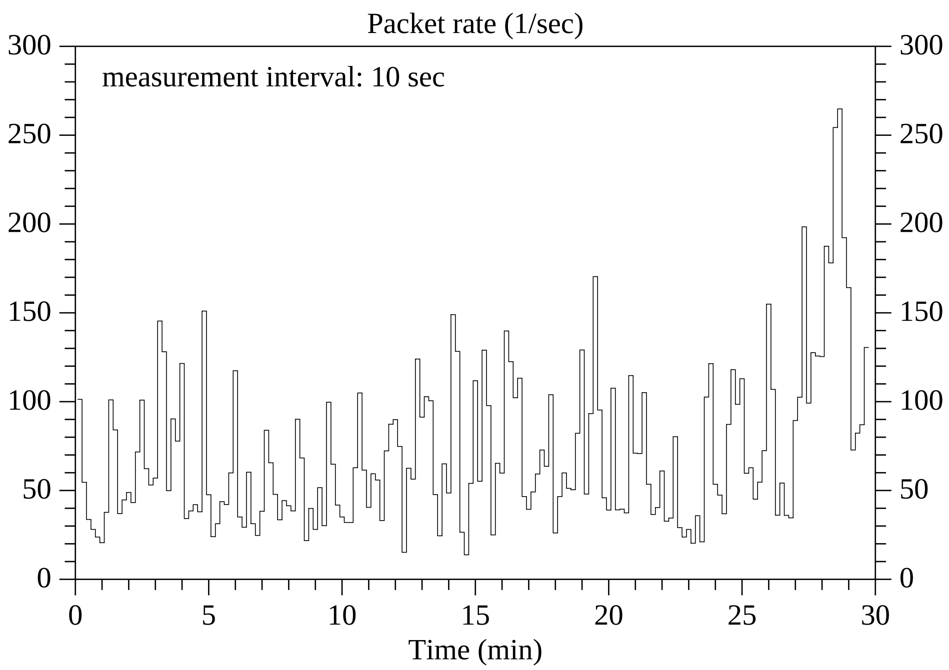


Fig. 1. Measured Packet Rate

As mentioned in the Introduction, the dynamic properties are typically related to the changes in traffic or load rate during some periods of time. In order to make the subject concrete, we give an example. Figure 1 shows the measured packet rate to and from a file server.

When the analyst examines a traffic rate, there are several dynamic properties that may be of interest. For example, the analyst may want to learn how much the traffic rate varies during one minute. In addition, the analyst may want to learn how much the expected traffic rate during one minute varies. We believe that such indices are useful in various situations of practical interest.

B. Theoretical Background: Spectral Representation and Filters

In spectral analysis of stationary stochastic processes a single realization $\{X_t\}$ is expressed as

$$X_t - \mu = \int_{-\pi}^{\pi} \exp(it\omega)\, dZ(\omega)\ , \tag{1}$$

where $\mu = \mathrm{E}(X_t)$ and $Z(\omega)$ is an orthogonal process on interval $(-\pi, \pi)$ such that

$$\mathrm{E}\left\{|dZ(\omega)|^2\right\} = h(\omega)\, d\omega\ . \tag{2}$$

The spectral density function $h(\omega)$ characterizes how $\mathrm{Var}(X_t)$ is distributed over different frequencies. When ω is close to 0, $h(\omega)$ gives the long-term variation and when ω close to π, $h(\omega)$ gives the short-term variation. The spectral density function provides a complete characterization of the variation in a stationary stochastic process. Unfortunately, the estimation of $h(\omega)$ is, in general, a difficult task; see e.g. [5; 6; 7; 8]. Therefore, we need an alternative characterization.

In signal processing the input is considered as a sum of components having different frequencies. Filters are used to pass some interval of frequencies while suppressing the other frequencies. A central concept of a filter is the “band-pass”. A low-pass filter passes only long-term variation and a high-pass filter passes only short-term variation.

Traditionally, linear filters such as moving averages and moving differences have been used. In recent years, nonlinear filters that are based on moving order statistics have become popular [9]. The reason is that these nonlinear filters obtain the gain close to ideal more easily and precisely than the linear filters to do.

The moving range,

$$r(X_t, k) = \max(X_{t-k}, \ldots, X_{t+k}) - \min(X_{t-k}, \ldots, X_{t+k}) \ , \tag{3}$$

provides a sequence containing mainly short-term variation. The moving average,

$$s(X_t, k) = \frac{1}{2k+1} \sum_{j=-k}^{k} X_{t+j} \ , \tag{4}$$

is the most widely used linear low-pass filter that removes the short-term variation and provides an estimate for the expected value of X_t.

In the example we were also interested in the variation of the expected traffic rate during one minute. This variation that we call the medium-term variation can be examined using the sequence of the moving ranges of the moving averages.

In the sequences of moving ranges and averages the original observations overlap in a similar fashion as in the method of overlapped batch means [10]. It is shown in [11] that the partial reuse of observations in consecutive batches is statistically efficient. Although the autocorrelation stronger, the sequence length is essentially longer.

C. Indices and Transforms

As stated in the Introduction, we characterize the dynamic properties in the original sequence by three indices: the steady-state mean, the average of short-term variation, and the average of medium-term variation. This implies the assumption that neither a trend nor a cyclic variation is present. The assumption is clearly restrictive, but we are currently not ready to implement general purpose trend and cycle detectors. The reason is that trends and cycles require special techniques that are extremely hard to automatize in a general case.

The detection of a trend is known as the startup problem in simulation. During the last two decades several algorithms for detecting a trend have been proposed. Most of them are either heuristic or based on the transient-behaviour in a single server queue. The survey [3] clearly indicates that further research is necessary. Especially, experimental studies should be conducted. The same is also true with the cyclic behaviour. In statistical literature various techniques have been proposed to detect and eliminate the seasonal variation; see e.g. [12]. We believe that the trend detector and the cycle detector require approaches that are different from each other and from our approach.

The indices that we estimate are based on the following three transforms of the original sequence $\{X_j\}_{j=1}^{N}$:

1. moving ranges with lag k_1 of the original sequence,

$$Y_{1,i} = \max(X_{i-k_1}, \ldots, X_{i+k_1}) - \min(X_{i-k_1}, \ldots, X_{i+k_1}) \,,\ i = 1 + k_1, \ldots, N - k_1$$

2. moving averages with lag k_2 of the original sequence,

$$Y_{2,i} = (X_{i-k_2} + \ldots + X_{i+k_2})/(2k_2 + 1) \,,\ i = 1 + k_2, \ldots, N - k_2$$

3. moving ranges with lag k_3 of the moving averages with lag k_2,

$$Y_{3,i} = \max(Y_{2,i-k_3}, \ldots, Y_{2,i+k_3}) - \min(Y_{2,i-k_3}, \ldots, Y_{2,i+k_3}) \,,\ i = 1 + k_2 + k_3, \ldots, N - (k_2 + k_3) \,.$$

We truncate the sequences so that we obtain a multivariate sequence

$$\{X_i, Y_{1,i}, Y_{3,i}\}_{i=1+k}^{N-k} \,, \tag{5}$$

where $k = \max(k_1, k_2 + k_3)$. The means that we estimate are:

$$\begin{aligned} \bar{x} &= \sum_{i=1+k}^{N-k} x_i/(N-2k) \,, \\ \bar{y}_1 &= \sum_{i=1+k}^{N-k} y_{1,i}/(N-2k) \,, \\ \bar{y}_3 &= \sum_{i=1+k}^{N-k} y_{3,i}/(N-2k) \,. \end{aligned} \tag{6}$$

Figure 2 demonstrates the basewindow of the user interface in our interactive simulation tool. The lags in this example are $k_1 = k_2 = k_3 = 3$. The basewindow provides a partial time series plot of the original output sequence and a glyph visualizing the triple $(\bar{x}, \bar{y}_1, \bar{y}_3)$:

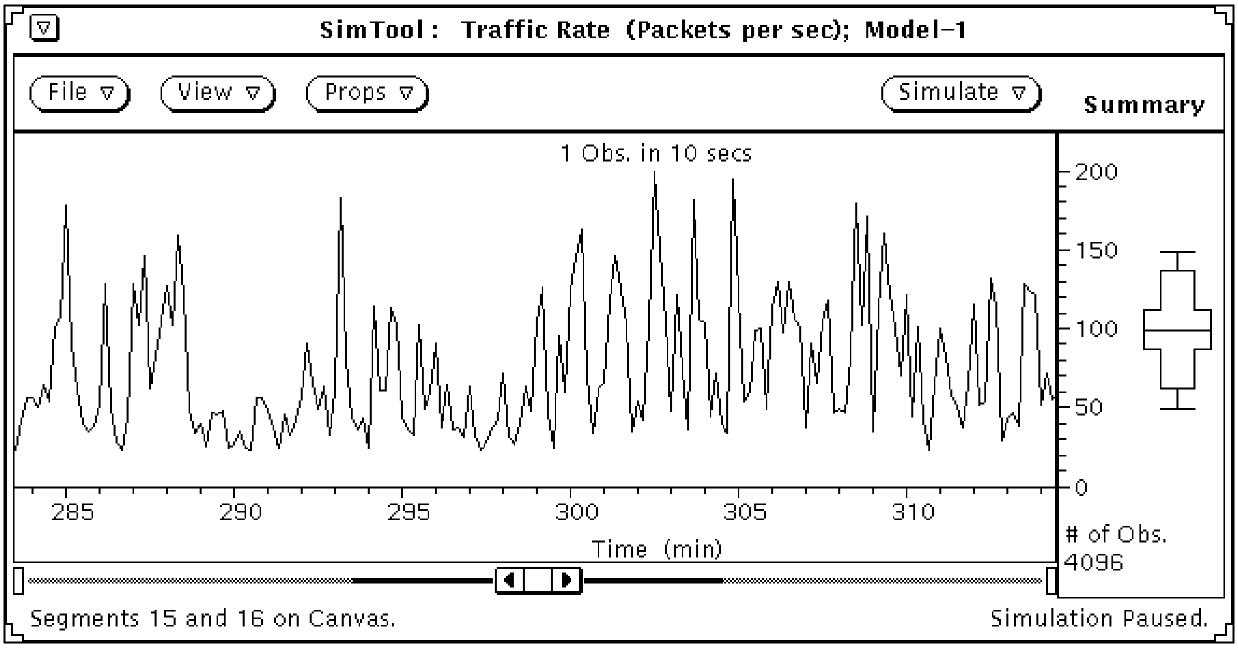
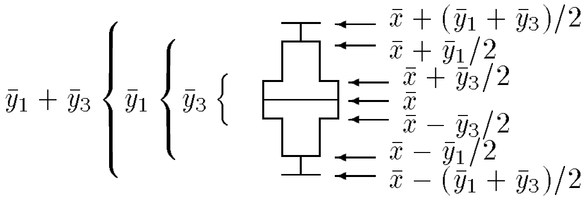


Fig. 2. Time Series Plot with Summary for Estimated Means



The boundaries between short-term, medium-term and long-term variation depend completely on the model examined and on objectives of the study. The terms are implied by the choice of the lags k_1, k_2, and k_3. Lag k_1 implies the short-term while lags k_2 and k_3 together imply the medium-term. Usually the lags can easily be determined from the objectives of the study and the sampling frequency.

III. Method for Assessing the Accuracy of Estimated Means

In this section we describe the method that we use in assessing the accuracy of estimated means. We start by outlining the traditional sequential procedure. We also propose how the procedures should be changed when the simulation is interactive. Next we describe the information provided to the user for assessing the accuracy. Finally, we consider the implementation details and introduce our choices.

A. Traditional Sequential Procedure and Interactive Simulation

Sequential procedures are widely used in simulation studies when a single mean is estimated. The analyst usually specifies the required accuracy for the estimate as the tolerance ε for the relative half-width of the $100(1-\alpha)$ % confidence interval.

The procedure estimates the confidence interval at prespecified checkpoints $n_1, n_2, \ldots, n_{max}$; usually $n_{j+1} = I\, n_j$, where I is a prespecified constant, usually $1 < I \leq 2$. The simulation is automatically terminated at the first checkpoint in which the relative half-width is less than ε. The estimation of confidence interval is usually based on the assumption that the distribution of $(\bar{X}(n) - \mu)/s_{\bar{X}(n)}$ is the Student t-distribution with $\nu(n)$ degrees of freedom. Therefore, the simulation is terminated at the first checkpoint in which

$$s_{\bar{X}(n)}/\bar{X}(n) \leq \varepsilon/t_{\nu(n)}(1-\alpha/2)\,, \tag{7}$$

where $\bar{X}(n) = \sum_{i=1}^{n} X_i/n$, $s^2_{\bar{X}(n)}$ is the estimated variance of $\bar{X}(n)$ with $\nu(n)$ degrees of freedom, and $t_{\nu(n)}(p)$ is the $100p^{\text{th}}$ percentile of the Student t-distribution with $\nu(n)$ degrees of freedom.

There are two points in the traditional sequential procedure that we want to change when the simulation is interactive. Firstly, we want to leave the decision of terminating the simulation run to the user. However, we provide information that corresponds to the relative half-width of the estimated confidence interval. Secondly, the run length usually grows exponentially: $n_{j+1} = I^j\, n_1$. This is inadequate in interactive simulation, since a new feedback to the user can only be delivered at the checkpoints. However, the rate of feedback to the user should be constant. Therefore, the run length should grow linearly: $n_{j+1} = n_j + n_{step}$.

B. Accuracy of Estimated Means

The relative half-width of the estimated confidence interval is the most widely used measure for the accuracy of an estimated mean. However, the relative half-width of the confidence interval is not a single number but a curve in a plane (see Figure 3). In principle, we could display the curve at each checkpoint. This is impractical especially when several means are estimated. Instead, we should ask the user to specify either the confidence levels or the relative half-widths. The response to the user will be either the estimated relative half-widths or the estimated confidence levels.

When we are simultaneously estimating several means, $\bar{X}_j(n)$, $j = 1, \ldots, m$, we must also consider the joint distribution of the estimated means. The Bonferroni inequality is the most widely used tool in simulation studies when several means are estimated; see e.g. [13, Ch. 4.8] or [2, Ch. 9.7]. When we estimate the confidence intervals, we ought to provide two relative half-widths

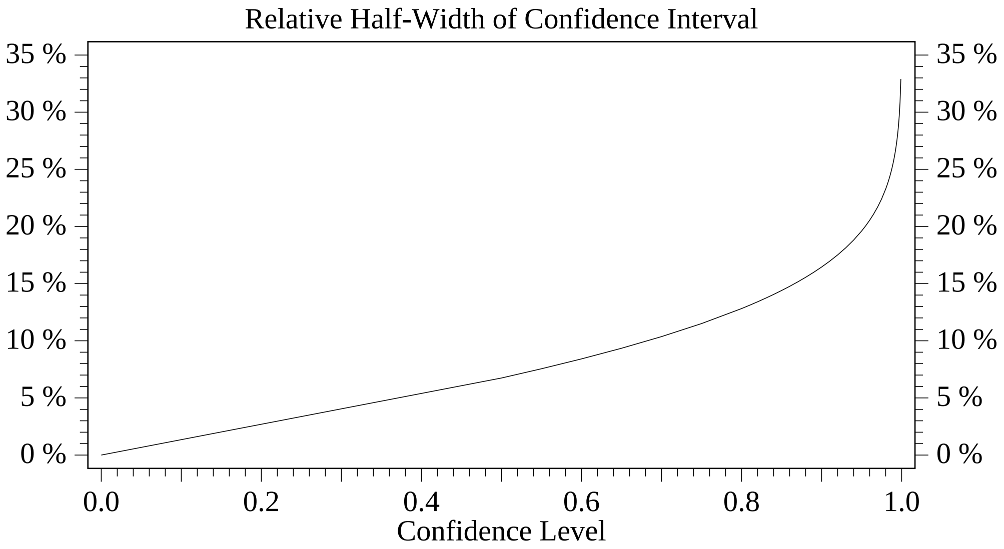


Fig. 3. An Example of Dependency Between the Relative Half-Width of the Confidence Interval and the Confidence Level

for each mean: $t_{\nu_j}(1-\alpha_j/2)s_{\bar{X}_j(n)}/\bar{X}_j(n)$ and $t_{\nu_j}(1-(\alpha_1+\ldots+\alpha_m)/2)s_{\bar{X}_j(n)}/\bar{X}_j(n)$. The first one is the relative half-width of the individual confidence interval and the second one is the relative half-width of the edge in the joint rectangular confidence area.

We have adopted another way of applying the Bonferroni inequality: Instead of estimating the relative half-widths for the given confidence levels, we estimate the confidence levels for the given relative half-widths. In other words, we ask the user to specify the tolerances for the relative half-widths, $\varepsilon_1, \ldots, \varepsilon_m$. The confidence levels of the intervals $I_j(n) = [\bar{X}_j(n)(1-\varepsilon_j), \bar{X}_j(n)(1+\varepsilon_j)]$ can be solved from the inequality (7):

$$1-\alpha_j \geq 1 - 2F_{\nu_j}(-\varepsilon_j \bar{X}_j(n)/s_{\bar{X}_j(n)})\,, \tag{8}$$

where $\bar{X}_j(n) = \sum_{i=1}^{n} X_{ij}/n$, $s^2_{\bar{X}_j(n)}$ is the estimated variance of $\bar{X}_j(n)$ with ν_j degrees of freedom and $F_\nu(t)$ is the cumulative distribution function of the Student t-distribution with ν degrees of freedom. Now we easily obtain using the Bonferroni inequality an estimated lower bound for the probability that each $\mu_j = \mathrm{E}(X_{\cdot j})$ is simultaneously on the interval $I_j(n)$:

$$\Pr\{\mu_1 \in I_1(n) \wedge \ldots \wedge \mu_m \in I_m(n)\} \geq 1 - \sum_{j=1}^{m} \alpha_j \geq 1 - 2\sum_{j=1}^{m} F_{\nu_j}(-\varepsilon_j \bar{X}_j(n)/s_{\bar{X}_j(n)})\,. \tag{9}$$

The traditional way of applying the Bonferroni inequality is based on the prespecified α_j's. In such a case the use of the Bonferroni inequality is practically limited to no more than 10 different parameters. However, our way of applying the Bonferroni inequality is different. In Eq. (9) the

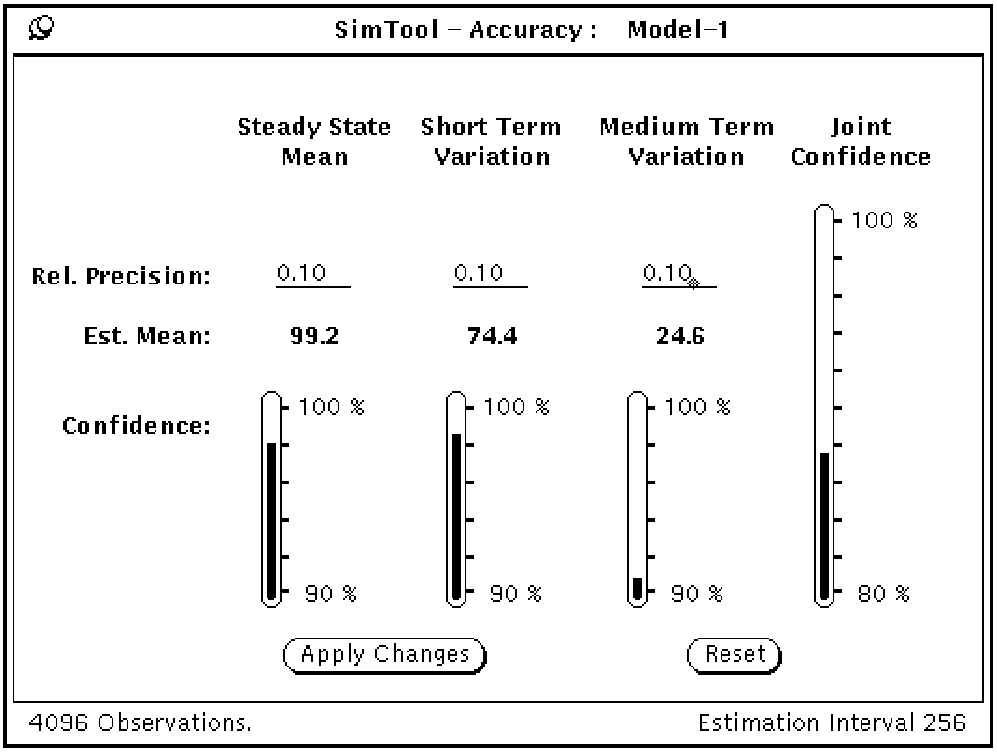


Fig. 4. Proposed Information for Assessing the Accuracy of Estimated Means

α_j's are not prespecified but estimated:

$$\alpha_j = 2F_{\nu_j}(-\varepsilon_j \bar{X}_j(n) / s_{\bar{X}_j(n)}) \ .$$

In this case the number of parameters that can practically be examined depends on the sequences examined, particularly on the factors $C_j = s_{\bar{X}_j(n)} / \varepsilon_j \bar{X}_j(n)$, $j = 1, \ldots, m$.

The rough approximation of $F_\nu(-t)$ by $a_\nu t^{-\nu}$ for large t's and small ν's [14, p. 949], can be used to exemplify the effects of the C_j. If C_1 is considerably larger than the others, then we have

$$\sum_{j=1}^{m} \alpha_j \approx 2a_{\nu_1} {C_1}^{\nu_1} \ .$$

In this case the effects of other sequences on the lower bound for the estimated joint confidence are negligible. On the hand, if all the C_j's are of the same magnitude then we have

$$\sum_{j=1}^{m} \alpha_j \approx 2ma_{\nu_1} {C_1}^{\nu_1} \ .$$

In this case our way of applying the Bonferroni inequality is practically equal to the traditional way. We have shown that our way of applying the Bonferroni inequality is asymptotically the most efficient way of applying the Bonferroni inequality [15].

Figure 4 demonstrates the accuracy information that our simulation tool provides to the user. The example is the same as before. We display the estimated confidences in four gauges, one for each individual mean and one for the joint confidence. When the user decides that the estimated confidences are high enough, the simulation can be terminated.

C. Implementation Considerations

The estimated confidence levels are based on the approximations $(\bar{X}_j(n) - \mu_j)/s_{\bar{X}_j(n)} \sim t_{\nu_j}$ and on the Bonferroni inequality. Hence we need an estimator for the variances of $\bar{X}_j(n)$. In principle any existing method that does not seriously underestimate the variance could be used. Our recommendation is the spectral method introduced in [16]. The spectral method is widely recognized as one of the most robust and reliable method for estimating the variance of a steady-state mean; see e.g. [3]. The most important advantage of the spectral method is that the batch means on which the estimated variance is based need not be uncorrelated.

The spectral method as described in [16] uses exponentially increasing sequence of checkpoints: $n_{j+1} = 1.5\, n_j$. However, we prefer linearly increasing sequence, because the accuracy information should be updated after constant time intervals. The linear growth makes the batching more complex. In addition, it requires that the whole output sequence must be available which is impractical. Therefore, we have done a compromise between the linear and exponential growth.

Our implementation of the spectral methods maintains up to $10M$ batches. When the number of batches grows to $10M$, we double the batch size and reduce the number of batches to $5M$ by averaging two adjacent batches. The confidence levels are estimated each time when M new batches are available, that is $M \times batch\_size$ new observations are generated. In order to make the spectral method computationally efficient, the available kM batch means are further temporarily reduced to M batch means before estimating the variances.

In our implementation of the spectral method a second-order polynomial is fitted to the first $K < M/4$ ordinates of the bias corrected logarithms of the smoothed periodogram. Periodogram ordinates are evaluated through the Fast Fourier Transform and the smoothing is done by averaging two adjacent ordinates. In the examples of Section IV.C we have used $M = 256$ and $K = 31$. The degrees of freedom in the variance estimate are 9. For details, see Appendix I.

IV. Evaluation of the Proposed Method

In this section we evaluate the proposed method for assessing the accuracy of simulation estimates. Firstly, we show that the method has a sound theoretical basis by showing that the estimators are asymptotically reliable. Secondly, we derive an approximation of the bias and report experimental results in a case of finite sample sizes. Finally, we apply the method in three examples.

When we are assessing the accuracy of simulation estimates, our parameters of interest are the probabilities that the unknown means μ_j , $j = 1, \ldots, m$, are both separately and simultaneously on the intervals $I_j(n) = [(1-\varepsilon_j)\bar{X}_j(n), (1+\varepsilon_j)\bar{X}_j(n)]$. When we approximate the distribution of $(\bar{X}_j(n) - \mu_j)/s_{\bar{X}_j(n)}$ by Student t-distributions with ν_j degrees of freedom, then the natural estimators for the parameters are

$$g(\hat{T}_j(n)) = 1 - 2F_{\nu_j}(-\varepsilon_j \hat{T}_j(n)), \ j = 1, \ldots, m, \tag{10}$$

and

$$g(\hat{T}_1(n), \ldots, \hat{T}_m(n)) = 1 - 2\sum_{j=1}^{m} F_{\nu_j}(-\varepsilon_j \hat{T}_j(n)), \tag{11}$$

where $\hat{T}_j(n) = \bar{X}_j(n)/s_{\bar{X}_j(n)}$.

A. Asymptotic Properties

The asymptotic properties of the estimators indicate how the estimators behave when the simulation run is long. We will show that our estimators are asymptotically reliable provided that the method for estimating the variances of the $\bar{X}_j(n)$'s provides a weakly consistent estimator for $\mathrm{Var}(\bar{X}_j(n))$. By reliability we mean that the true probability of $\{\mu_j \in I_j(n)\}$ is at least the estimated confidence. We first consider the individual confidences and then the joint confidence.

Define

$$N_j = \text{smallest } n \geq n_0 \text{ such that } g(\hat{T}_j(n)) \geq 1 - \alpha_j \, . \tag{12}$$

The definition of N_j in (12) is equivalent to the definition

$$N_j = \text{smallest } n \geq n_0 \text{ such that } s_{\bar{X}_j(n)} / \bar{X}_j(n) \leq \varepsilon_j / t_{\nu_j}(1 - \alpha_j/2), \tag{13}$$

which is the same termination rule as in the traditional sequential procedure. It is shown in [17] that the definition of N_j in (13) provides an asymptotically consistent confidence interval, that is

$$\lim_{\varepsilon_j \to 0} \Pr\{|\bar{X}_j(n) - \mu_j| \leq \varepsilon_j |\bar{X}_j(n)|\} = 1 - \alpha_j \, , \tag{14}$$

if the sequence $\{X_{ij}\}_{i=1}^{\infty}$ is covariance stationary and if $s^2_{\bar{X}_j(n)}$ is a strongly consistent estimator of $\mathrm{Var}(\bar{X}_j(n))$. The strong consistency means that $\nu_j s^2_{\bar{X}_j(n)} / \mathrm{Var}(\bar{X}_j(n))$ converges in distribution to χ^2-distribution with ν_j degrees of freedom and $\nu_j \to \infty$ as $n \to \infty$. Since the definitions of N_j in (12) and in (13) are equivalent, the estimated confidences are asymptotically consistent.

Most of the methods for estimating $\mathrm{Var}(\bar{X}_j(n))$ that are currently used do not satisfy the condition of $\nu_j \to \infty$ as $n \to \infty$. However, if $s^2_{\bar{X}_j(n)}$ is weakly consistent, that is it converges (almost surely) to $\mathrm{Var}(\bar{X}_j(n))$ as $n \to \infty$, and if we approximate the distribution of $(\nu_j s^2_{\bar{X}_j(n)})/(\mathrm{Var}(\bar{X}_j(n)))$ by the χ^2-distribution with constant (ν_j) degrees of freedom, then the definitions of N_j in (12) and in (13) provide confidence intervals that satisfy

$$\lim_{\varepsilon_j \to 0} \Pr\{|\bar{X}_j(n) - \mu_j| \leq \varepsilon_j |\bar{X}_j(n)|\} \geq 1 - \alpha_j \,. \tag{15}$$

Therefore, we can conclude that the estimated probabilities of $\{\mu_j \in I_j(n)\}$ are asymptotically reliable, when the method for estimating the variance of the mean is weakly consistent.

Let us now consider the joint confidence. Define

$$N = \text{smallest } n \geq n_0 \text{ such that } g(\hat{T}_1(n), \ldots, \hat{T}_m(n)) \geq 1 - \alpha \,. \tag{16}$$

The definition of N in (16) implies that we choose the α_j's such that $\sum_{j=1}^{m} \alpha_j = \alpha$ and

$$g(\hat{T}_j(N)) \geq 1 - \alpha_j > g(\hat{T}_j(N-1)) \,, \; j = 1, \ldots, m \,. \tag{17}$$

Using the Bonferroni inequality we obtain

$$1 - \alpha = 1 - \sum_{j=1}^{m} \alpha_j \leq \lim_{\varepsilon_1, \ldots, \varepsilon_m \to 0} \Pr\{\mu_1 \in I_1(N) \wedge \ldots \wedge \mu_m \in I_m(N)\} \,. \tag{18}$$

Therefore, the estimated joint confidence is asymptotically reliable provided that the method for estimating the variances of the means is weakly consistent.

B. Finite Sample Properties

The asymptotic reliability indicates that the method is reliable when the ε_j's are small. However, it does not give any indication how small the ε's should be. Now we consider the finite sample properties of an estimated confidence

$$g(\hat{T}) = 1 - 2F_\nu(-\varepsilon \hat{T}) \,, \tag{19}$$

where $\hat{T} = \bar{X}/s_{\bar{X}}$.

Since $g(\hat{T})$ is a non-linear function of $\hat{T}$, $g(\hat{T})$ is usually a biased estimator for $g(T)$, $T = \mathrm{E}(\bar{X})/\,\mathrm{Var}(\bar{X})$. Using the Taylor series representation of $g(\hat{T})$,

$$g(\hat{T}) = g(T) + \sum_{k=1}^{\infty} \frac{g^{(k)}(T)}{k!} (\hat{T} - T)^k \,, \tag{20}$$

we obtain approximations for the bias of $g(\hat{T})$, $b(t) = \mathrm{E}[g(\hat{T})] - g(T)$.

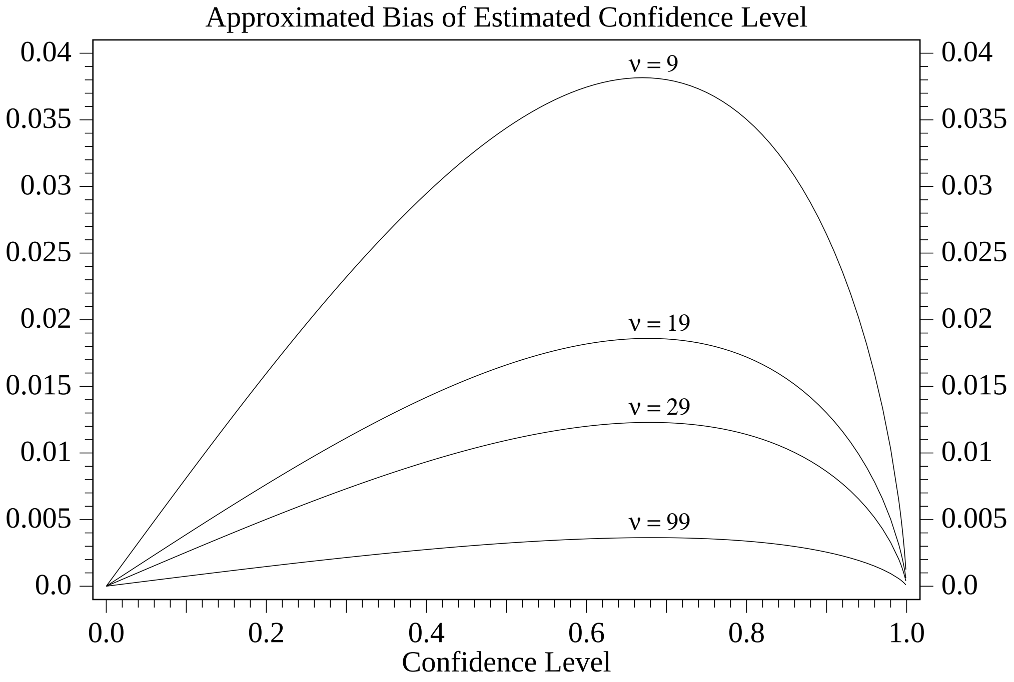


Fig. 5. Approximated Bias of Estimated Confidence Level

When $(\bar{X} - \mu)/s_{\bar{X}} \sim t_\nu$, then $\hat{T}$ has the non-central t-distribution with the non-centrality parameter $\mathrm{E}(\bar{X})/\sqrt{\mathrm{Var}(\bar{X})}$. Therefore,

$$b(g(\hat{T})) \approx \frac{3}{2\nu}\,\varepsilon T\, f_\nu(-\varepsilon T)\,. \tag{21}$$

Figure 5 shows the approximated bias as the function of confidence level for $\nu = 9$, 19, 29, and 99. The figure indicates that the estimated confidence is greater than the true confidence but the bias is not serious for large ν's. In addition, for confidences greater than 0.9, the bias is small even with moderate ν's.

We also examined the bias of estimated confidences experimentally. We generated 10 000 sequences of independent observations from the unit exponential distribution and estimated the probabilities of $\{(1-\varepsilon)\bar{X}(n) \le 1 \le (1+\varepsilon)\bar{X}(n)\}$ for $n = 10, 20, \ldots, 100, 200, \ldots, 1000$ and $\varepsilon = 0.25$, 0.2, 0.15, 0.125, 0.1, 0.075, 0.05, and 0.025:

$$\Pr\{(1-\varepsilon)\bar{X}(n) \le 1 \le (1+\varepsilon)\bar{X}(n)\} = \int_{1/(1+\varepsilon)}^{1/(1-\varepsilon)} \frac{n^n}{\Gamma(n)} x^{n-1} \exp(-nx)\, dx\,, \tag{22}$$

and

$$\widehat{\Pr}\{(1-\varepsilon)\bar{X}(n) \le 1 \le (1+\varepsilon)\bar{X}(n)\} = 1 - 2F_{n-1}(-\varepsilon\bar{X}/s_{\bar{X}})\,, \tag{23}$$

where $F_{n-1}(x)$ is the cumulative distribution function of the Student t-distribution with $n-1$ degrees of freedom and $s_{\bar{X}}^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n^2-n)$.

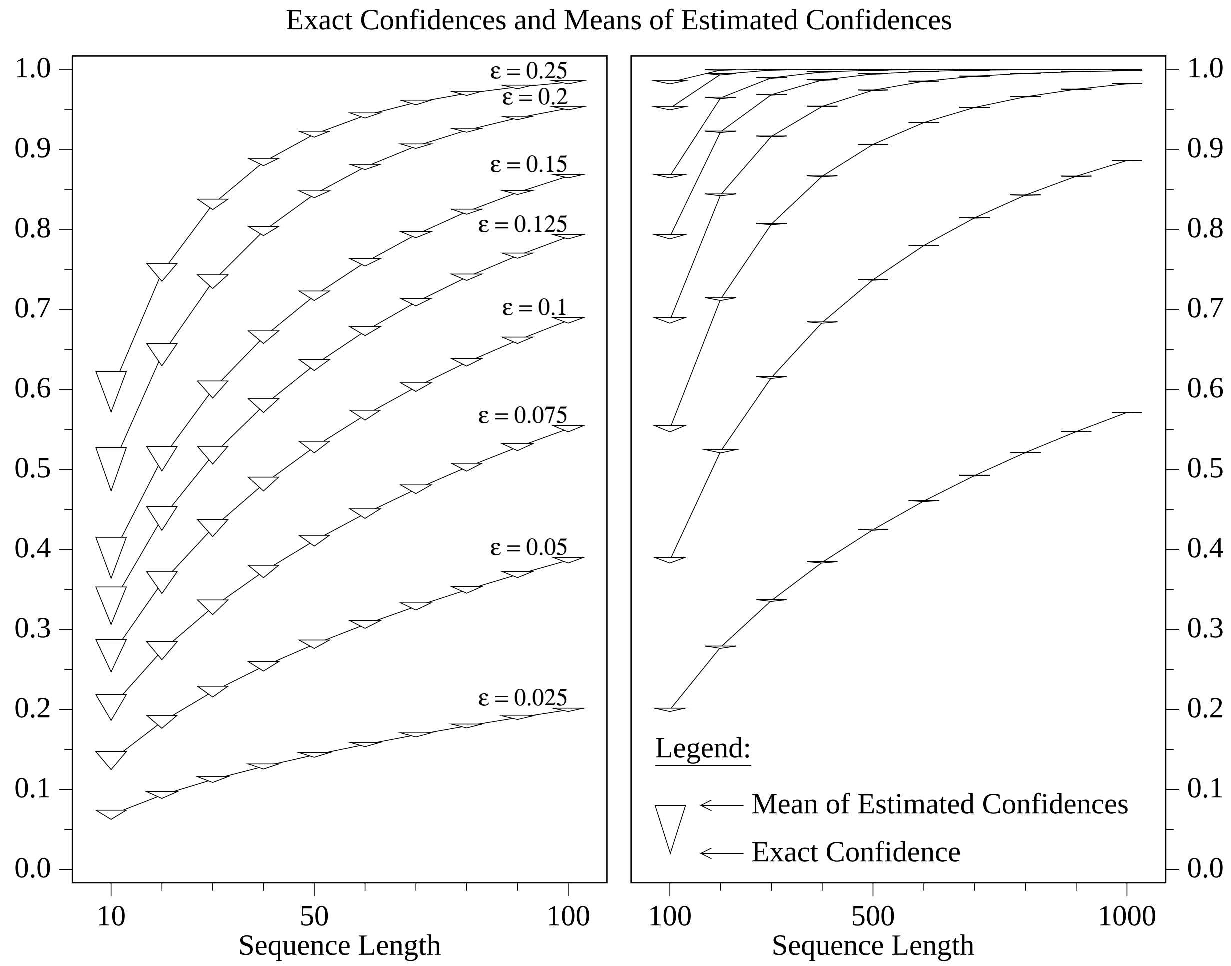


Fig. 6. Exact and Means of Estimated Probabilities of $\{|\bar{X} - 1| \leq \varepsilon|\bar{X}|\}$

$X \sim \mathrm{Exp}(1)$; 10 000 independent replications

The example demonstrates a common practical situation in which the distribution of the observations is skewed. The skewness decreases the rate with which $(\bar{X} - \mu)/s_{\bar{X}}$ converges to the normal limiting distribution. Therefore, the error due to the approximation $(\bar{X} - \mu)/S_{\bar{X}} \sim t_{n-1}$ affects the bias when n is small.

Figure 6 shows the exact and means of estimated confidences. The figure is divided into two parts. On the left-hand side we show the sequence lengths $10, 20, \ldots, 100$ and on the right-hand side the sequence lengths $100, 200, \ldots, 1000$. The exact probability and the mean of estimated confidence for a pair (n, ε) is visualized as a triangle: the tip indicates the location of the exact confidence and the base the location of the mean of estimated confidences. Therefore, the height of the triangle shows the size of the bias. The figure clearly confirms the conclusion based on the approximated bias. When $\varepsilon \leq 0.2$ and the confidence is at least 0.9, then the bias is negligible. The figure also shows that the bias decreases smoothly as the sequence length increases.

C. Example Applications

As examples of estimating the dynamic properties in queueing systems, we simulated three different models: an M/M/1-queue, an M/D/1-queue, and an imbedded autoregressive process.

In the M/M/1-queue with arrival rate $\lambda = 1$ and service rate $\mu = 1.25$ we examined the queue length. The sampling interval was 1. In the M/D/1-queue with $\lambda = 125$ and a constant service time $S = 0.002$ we examined the server utilization. The sampling interval was 1. The imbedded AR-process examined models the packet rate to and from a file server. The model is based on measurement data. The derivation of the imbedded AR-process is described in Appendix II.

The generated sequences $\{X_i\}_{i=1}^{N}$ were transformed into the three dimensional sequences

$$\{X_i, Y_{1,i}, Y_{3,i}\}_{i=1+2k}^{N-2k}, \tag{24}$$

where $k_1 = k_2 = k_3 = k$:

$$\begin{aligned} Y_{1,i} &= \max(X_{i-k}, \ldots, X_{i+k}) - \min(X_{i-k}, \ldots, X_{i+k}), && (25) \\ Y_{2,i} &= (X_{i-k} + \ldots + X_{i+k})/(2k+1), && (26) \\ Y_{3,i} &= \max(Y_{2,i-k}, \ldots, Y_{2,i+k}) - \min(Y_{2,i-k}, \ldots, Y_{2,i+k}). && (27) \end{aligned}$$

In the M/M/1- and M/D/1-queue $k = 5$ and in the imbedded ARP $k = 3$ were used.

Although our method for assessing the accuracy is intended for interactive simulation, we applied it as a termination rule. The simulation run is terminated as soon as the estimated confidence is at least the requested confidence for the given relative half-width. The requested confidences examined were 0.90, 0.95, and 0.99; and the relative half-widths, $\varepsilon_X = \varepsilon_{Y_1} = \varepsilon_{Y_3} = \varepsilon =$ 0.20, 0.10, and 0.05.

The simulation experiment was repeated 1 000 times with each model. We estimated both the individual confidences and the joint confidence. The results are summarized in Tables 1 and 2. Table 1 gives the observed confidences and Table 2 the means of run lengths.

The observed confidence, that is the fraction of runs in which the relative distance from the estimated mean to the exact mean is less than the given tolerance (ε), exceeds the requested confidence $(1-\alpha)$ in most of the cases. When the Bonferroni inequality is used this is not surprising since the inequality provides a lower bound.

When the experiment is independently repeated, we can use the binomial distribution and the observed confidences in testing the hypothesis that the true (unknown) confidence is high enough. We conclude that the termination rule works satisfactorily for the triple $(model, \varepsilon, \alpha)$ if the null-hypothesis H_0 can not be rejected at significance level α in the statistical test of

$$H_0: \ \text{true confidence} \ \geq 1 - \alpha \qquad vs.$$

Table 1. Observed Confidences

	Requested Confidence								
	0.90			0.95			0.99		
	ε			ε			ε		
	0.20	0.10	0.05	0.20	0.10	0.05	0.20	0.10	0.05
M/M/1-queue									
Steady-state mean	.831†	.883†	.878†	.887†	.941	.953	.972	.987	.995
Short-term variation	.949	.930	.943	.980	.972	.975	.997	.994	.993
Medium-term variation	.915	.925	.925	.971	.965	.964	.994	.995	.994
All three simultaneously	.867†	.899	.891	.910	.944	.956	.978	.989	.996
M/D/1-queue									
Steady-state mean	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Short-term variation	.999	.987	.959	.999	.995	.973	1.00	1.00	.996
Medium-term variation	.985	.953	.928	.992	.986	.972	.998	.997	.997
All three simultaneously	.984	.978	.947	.995	.990	.980	1.00	.999	.998
Imbedded ARP									
Steady-state mean	.967	.924	.938	.991	.968	.968	1.00	.994	.994
Short-term variation	.994	.945	.941	.999	.982	.970	1.00	.996	.997
Medium-term variation	.945	.935	.919	.986	.967	.966	1.00	.998	.994
All three simultaneously	.977	.960	.966	.997	.985	.988	1.00	.999	.999

†: Hypothesis of acceptable confidence rejected.

$$H_A: \quad \text{true confidence} \quad < 1-\alpha \ ,$$

where $1-\alpha$ is the requested confidence. This is a modification of the approach used in [18] to evaluate the observed coverage.

The null-hypothesis of acceptable confidence can only be rejected in five cases. These cases are indicated by '†' in Table 1. All rejections occurred in the M/M/1-queue. In four cases the steady-state mean was estimated and in one case all three means were estimated. The test points $(\varepsilon, 1-\alpha)$ were $(0.20, 0.90)$, $(0.20, 0.95)$, $(0.20, 0.99)$, and $(0.10, 0.90)$. An obvious explanation is that in these cases the run length is not long enough for the t-approximation to be appropriate; see also [19]. The overall conclusion is that the proposed method provides reliable estimates for the confidences.

The means of run lengths clearly demonstrates the advantage of our way applying the Bonferroni inequality. When the run length needed to obtain an accurate estimate for one mean is essentially longer than the run lengths for the others, only a few additional observations are needed to obtain simultaneously accurate estimates for all the means.

Table 2. Means of Run Lengths

	Requested Confidence								
	0.90			0.95			0.99		
	ε			ε			ε		
	0.20	0.10	0.05	0.20	0.10	0.05	0.20	0.10	0.05
M/M/1-queue									
Steady-state mean	6,688	29,233	129,473	10,436	46,796	200,450	22,502	101,018	424,641
Short-term variation	482	1,548	6,778	636	2,450	10,135	1,191	5,297	20,347
Medium-term variation	1,191	5,834	21,985	2,042	8,852	33,102	4,563	17,313	67,377
All three simultaneously	7,323	30,607	133,933	10,907	47,809	203,944	22,810	101,843	426,983
M/D/1-queue									
Steady-state mean	256	256	256	256	256	256	256	256	259
Short-term variation	262	563	1,078	295	696	1,497	483	927	2,926
Medium-term variation	322	902	2,719	417	1,264	4,093	710	2,175	8,457
All three simultaneously	368	1,055	3,076	479	1,359	4,490	815	2,266	8,958
Imbedded ARP									
Steady-state mean	550	1,528	5,946	863	2,405	9,103	1,386	4,543	18,271
Short-term variation	506	894	3,234	765	1,378	5,020	1,054	2,562	9,951
Medium-term variation	650	1,948	7,274	1,036	2,888	11,404	1,647	5,629	22,308
All three simultaneously	887	2,890	11,434	1,257	4,076	16,149	2,021	7,358	29,441

V. Summary

We have proposed a way of characterizing and estimating the dynamic properties of a queueing system. Our focus has been on interactive simulation. The proposed way of assessing the accuracy of estimates in interactive simulation is shown to be reliable. The proposed method has two important advantages. Firstly, we can use the Bonferroni inequality in the most efficient way. Secondly, since we translate the problem of estimating the dynamic properties into the problem of estimating several means simultaneously, we can apply an existing method for estimating the variances of means.

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Appendix I: Spectral Method for Estimating the Variance of Mean

The spectral method for estimating the variance of the mean was introduced in [16]. The method assumes that the output sequence $\{X_j\}_{j=1}^N$ is covariance stationary, i.e. $\mathrm{E}(X_j) = \mu$ and $\mathrm{Cov}(X_j, X_{j+k}) = \gamma_k$. Under these assumptions

$$\lim_{N\to\infty} N \ \mathrm{Var}[\bar{X}(N)] = h(0)\,,$$

where $\bar{X}(N) = \sum_{j=1}^N X_j/N$ and $h(0)$ is the spectral density of the sequence $\{X_j\}_{j=1}^N$ at zero frequency. Therefore as N becomes large, we can approximate the variance of the mean by

$$\mathrm{Var}(\bar{X}(N)) \approx h(0)/N \ .$$

It is shown in [16] that the sequence of batch means $\{\bar{X}_k\}_{k=1}^M$,

$$\bar{X}_k = \sum_{j=1}^{B} X_{(k-1)B+j}/B\,,\ k = 1,\ldots,M \text{ and } M\,B = N\,,$$

can be used instead of the original sequence, since

$$h_B(0) = h(0)/B\,,$$

where $h_B(0)$ is the spectral density of the sequence $\{\bar{X}_k\}_{k=1}^M$ at zero frequency. Therefore, $\mathrm{Var}(\bar{X}(N))$ can be estimated by $\hat{h}_B(0)/M$.

The estimate of $h_B(0)$ is based on the periodogram $A(m/M)$ defined by

$$A(m/M) = \left| \sum_{k=1}^{M} \bar{X}_k \exp\left(-2\pi i \frac{(k-1)m}{M}\right)\right|^2 \Big/ M$$

where $i = \sqrt{-1}$. It can be efficiently obtained through the discrete Fourier transform of $\{\bar{X}_k\}_{k=1}^M$. Under very general conditions, see e.g. [20], the periodogram ordinates have the following approximate properties

$$\begin{array}{ll} \mathrm{E}[A(m/M)] \approx h_B(m/M) & 0 < m < M/2\,, \\ \mathrm{Var}[A(m/M)] \approx h_B^2(m/M) & 0 < m < M/2\,, \\ \mathrm{Cov}[A(m_1/M), A(m_2/M)] \approx 0 & 0 < m_1 \neq m_2 < M/2\,. \end{array}$$

Asymptotically periodogram ordinates are distributed as multiples of independent χ^2 random variables with two degrees of freedom. At $m = 0$ we have $A(0) = M\bar{X}^2$ that does not contain any information about $h_B(0)$. On the other hand, the other ordinates do not depend on the mean. Therefore, we base our estimator of $h_B(0)$ on regression.

There are two main problems associated with the application of regression techniques to the periodogram. Firstly, the variance is not constant. Secondly, the χ^2 distribution with two degrees of freedom is skewed. The logarithmic transformation is a well-known technique to solve the problem of variance heterogeneity, see [21]. The skewness can be reduced by taking the average of two adjacent ordinates.

Let $\omega_m = (4m-1)/2M$. The average of two adjacent ordinates provides an estimate for $h_B(\omega_m)$, $m = 1, \ldots, K < M/4$:

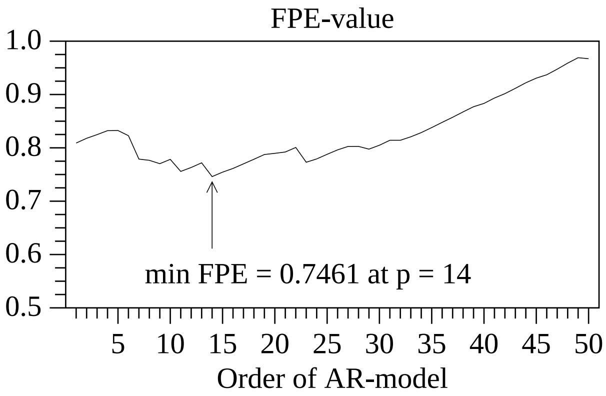
$$\hat{h}_B(\omega_m) = \frac{A((2m-1)/M) + A(2m/M)}{2}\,.$$

The distribution of $\hat{h}_B(\omega_m)/h_B(\omega_m)$ is approximately a multiple of the χ^2 distribution with four degrees of freedom.

We base our estimate of $h_B(0)$ on the regression model $\mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \boldsymbol{\epsilon}$, where

$$y_m = \log\left(\hat{h}_B(\omega_m)\right) + \mu_1, \qquad \mathbf{X} = \begin{pmatrix} 1 & \omega_1 & \omega_1^2 \\ \vdots & \vdots & \vdots \\ 1 & \omega_K & \omega_K^2 \end{pmatrix},$$

and the ϵ_m's are independent and identically distributed: $\mathrm{E}(\epsilon_m) = 0$ and $\mathrm{E}({\epsilon_m}^k) = \mu_k$, $k > 1$.



k	β_k	k	β_k	k	β_k
0	.6373	5	.0757	10	.0396
1	.4060	6	.0603	11	-.2236
2	-.0416	7	.2318	12	.0465
3	.0847	8	-.0641	13	-.0860
4	.0570	9	-.1398	14	.2103

Fig. 7. Summary of the Identified Imbedded Autoregressive Process

When $\hat{h}_B(\omega_m)/h_B(\omega_m)$ is regarded as a multiple of a χ^2 random variable with ν degrees of freedom, the first four μ's are, see [21]:

$$\mu_1 \approx \frac{1}{\nu} + \frac{1}{3\nu^2}\,,\ \mu_2 \approx \frac{2}{\nu-1}\,,\ \mu_3 \approx -\left(\frac{2}{\nu-1}\right)^2\,,\ \mu_4 \approx \left(\frac{2}{\nu-1}\right)^3 + 3\left(\frac{2}{\nu-1}\right)^2\,.$$

The least square estimate of $\boldsymbol{\alpha}$ is given by $\hat{\boldsymbol{\alpha}} = \mathbf{C}\mathbf{y} = \boldsymbol{\alpha} + \mathbf{C}\boldsymbol{\epsilon}$, where $\mathbf{C} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Hence $\mathrm{E}[(\hat{\alpha}_0 - \alpha_0)^k] = {\mu'}_k = \mu_k \sum_{j=1}^{K} {c_{1j}}^k$.

Our parameter of interest is $\exp(\alpha_0)$, the natural estimate of which is $\exp(\hat{\alpha}_0)$. In general $g(\hat{T})$ is a biased estimate of $g(T)$, but the bias can be reduced; see e.g. [22, p. 260]. Using Taylor expansion $\exp(\hat{\alpha}_0)$ can be written as

$$\exp(\hat{\alpha}_0) = \exp(\alpha_0) \sum_{k=0}^{\infty} \frac{(\hat{\alpha}_0 - \alpha_0)^k}{k!}\ .$$

Hence an approximately unbiased estimate of $h_B(0)$ is obtained as

$$\hat{h}_B(0) = \exp(\hat{\alpha}_0)/(1 + {\mu'}_2/2 + {\mu'}_3/6 + {\mu'}_4/24).$$

When a confidence interval is constructed using normal approximation, the degrees of freedom in the variance estimate is needed. Since we consider $\hat{h}_B(0)/h_B(0)$ as a multiple of a χ^2 variate, the degrees of freedom are $2\,\mathrm{E}\{\chi^2\}^2/\,\mathrm{Var}(\chi^2)$. Since $\hat{h}_B(0)/h_B(0) = (\sum(\hat{\alpha}_0 - \alpha_0)^k/k!)/(1 + {\mu'}_2/2 + {\mu'}_3/6 + {\mu'}_4/24)$, the degrees of freedom are about $3/(3{\mu'}_2 + 2{\mu'}_3 + {\mu'}_4)$.

An algorithm implementing this spectral method can be found in [3].

Appendix II: Imbedded Autoregressive Process

The imbedded autoregressive process used as the example in Section IV.C was constructed in the following way.

1) Identification of the imbedded autoregressive process consists of three phases.

In the first phase the measured sequence $\{x_j\}_{j=1}^n$ is replaced by the normal scores $\{z_j\}_{j=1}^n$: If the k^{th} order statistic is x_j, i.e. $x_j = x_{(k)}$, then $z_j = \Phi^{-1}(\frac{k}{n+1})$, where $\Phi(z)$ is the standard normal distribution function.

In the second phase, the order p of the autoregressive model, $Z_t = \sum_{k=1}^p \beta_k Z_{t-k} + \beta_0 \epsilon_t$, is determined using Akaike's final prediction error (FPE) criterion [23].

In the third phase the NAG Library routine G13ACF [24] is used to estimate the parameters $\beta_0, \ldots, \beta_p$. Figure 7 gives the parameters and shows the FPE-values.

2) Generation of the normal autoregressive process $\{Z_t\}$'s is done using the NAG Library routines G05EGF (setup routine) and G05EWF (generation routine). The imbedded AR-process $\{X_t\}$ was constructed using the transform:

$$X_t = F_X^{-1}[\Phi(Z_t)]\,,$$

where

$$F_X^{-1}(p) = x_{(j-1)}[j - p(n+1)] + x_{(j)}[p(n+1) - j + 1]\,;$$

when

$$\frac{j-1}{n+1} \le p \le \frac{j}{n+1}\,,\; j = 1, \ldots, n+1\,,$$

and $x_{(0)}$ and $x_{(n+1)}$ are interpolated:

$$\begin{aligned} x_{(0)} &= 2x_{(1)} - x_{(2)}\,, \\ x_{(n+1)} &= 2x_{(n)} - x_{(n-1)}\,. \end{aligned}$$