Three Concepts: Information

Lecture 3: Source Coding: Theory

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UNIVERSITY OF HELSINKI

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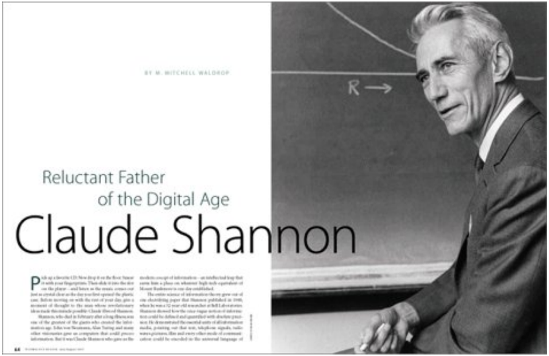
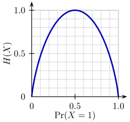


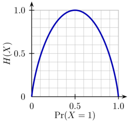
"The real birth of modern information theory can be traced to the publication in 1948 of Claude Shannon's *"The Mathematical Theory of Communication"* in the Bell System Technical Journal. "
(Encyclopædia Britannica)

1. Entropy and Information
   - Entropy
   - Information Inequality
   - Data Processing Inequality



1. Entropy and Information
   - Entropy
   - Information Inequality
   - Data Processing Inequality

2. Data Compression
   - Asymptotic Equipartition Property (AEP)
   - Typical Sets
   - Noiseless Source Coding Theorem



Entropy

Given a discrete random variable X with pmf p_X, we can measure the amount of "surprise" associated with each outcome $x \in \mathcal{X}$ by the quantity

$$I_X(x) = \log_2 \frac{1}{p_X(x)} \quad .$$

The less likely an outcome is, the more surprised we are to observe it. (The point in the log-scale will become clear shortly.)

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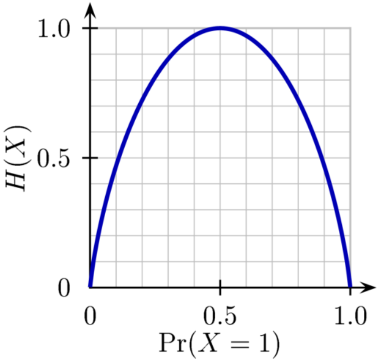
The **entropy** of X measures the *expected* amount of "surprise":

$$H(X) = E[I_X(X)] = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{1}{p_X(x)} \quad .$$

Binary Entropy Function

For binary-valued X, with $p = p_X(1) = 1 - p_X(0)$, we have

$$H(X) = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p} \ .$$



More Entropies

1. the **joint entropy** of two (or more) random variables:

$$H(X,Y) = \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} p_{X,Y}(x,y) \log_2 \frac{1}{p_{X,Y}(x,y)} \ ,$$

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2. the **entropy of a conditional distribution**:

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The conditional entropy $H(X \mid Y)$ measures the *expected* uncertainty about X when the value Y is known.

Chain Rule of Entropy

Remember the chain rule of probability:

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Proof.

$$\log_2 \frac{1}{p_{X,Y}(x, y)} = \log_2 \frac{1}{p_Y(y)} + \log_2 \frac{1}{p_{X|Y}(x \mid y)}$$

$$\Leftrightarrow E\left[\log_2 \frac{1}{p_{X,Y}(x, y)}\right] = E\left[\log_2 \frac{1}{p_Y(y)}\right] + E\left[\log_2 \frac{1}{p_{X|Y}(x \mid y)}\right]$$

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Logarithmic scale makes entropy **additive**.

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measures the average decrease in uncertainty about X when the value of Y becomes known.

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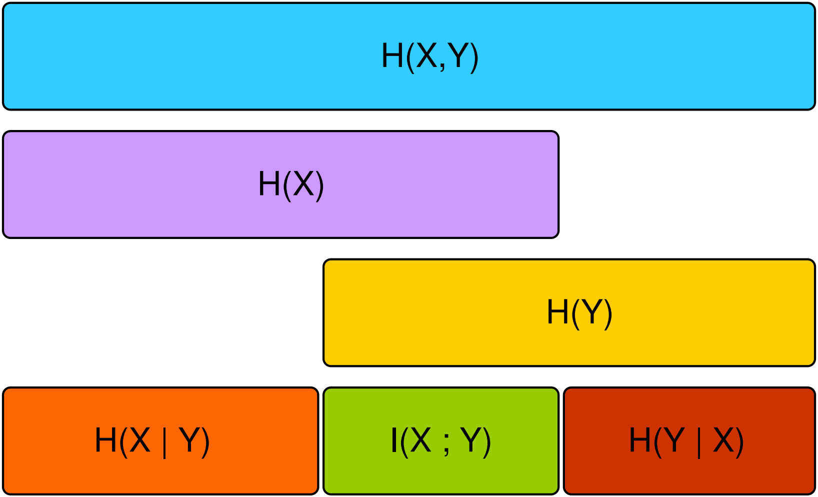
On the average, X gives as much information about Y as Y gives about X.

Relationships between Entropies



Information Inequality

Kullback-Leibler Divergence

The *relative entropy* or **Kullback-Leibler divergence** between (discrete) distributions p_X and q_X is defined as

$$D(p_X \parallel q_X) = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{p_X(x)}{q_X(x)} \quad .$$

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(We consider $p_X(x) \log_2 \frac{p_X(x)}{q_X(x)} = 0$ whenever $p_X(x) = 0$.)

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For any two (discrete) distributions p_X and q_X, we have

$$D(p_X \parallel q_X) \geq 0$$

with equality iff $p_X(x) = q_X(x)$ for all $x \in \mathcal{X}$.

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Proof. Gibbs!

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Proof.

$$\begin{aligned} I(X\,;\,Y) &= H(X) - H(X \mid Y) \\ &= H(X) + H(Y) - H(X, Y) \\ &= \sum_{\substack{x\in\mathcal{X}\\ y\in\mathcal{Y}}} p_{X,Y}(x,y)\, \log_2 \frac{p_{X,Y}(x,y)}{p_X(x)\, p_Y(y)} \\ &= D(p_{X,Y} \parallel p_X p_Y) \geq 0\ . \end{aligned}$$

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In addition, $D(p_{X,Y} \parallel p_X p_Y) = 0$ iff $p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$ for all $x \in \mathcal{X}, y \in \mathcal{Y}$. This means that variables X and Y are *independent* iff $I(X\,;\,Y) = 0$.

Properties of Entropy

Properties of entropy:

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   Proof. Let $u_X(x) = \frac{1}{|\mathcal{X}|}$ be the uniform distribution over $\mathcal{X}$.

$$0 \leq D(p_X \parallel u_X) = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{p_X(x)}{u_X(x)} = \log_2 |\mathcal{X}| - H(X) \ .$$

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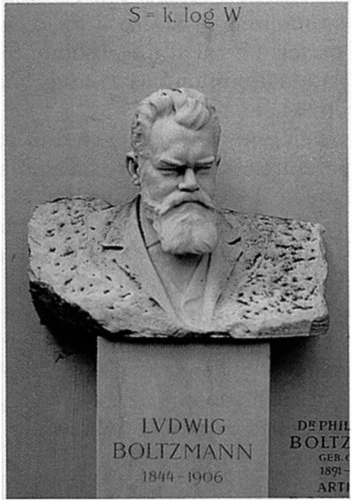
   A **combinatorial** approach to the definition of information (Boltzmann, 1896; Hartley, 1928; Kolmogorov, 1965):

$$S = k \ln W \quad .$$

Ludvig Boltzmann (1844–1906)



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   On the average, knowing another r.v. can only reduce uncertainty about X. However, note that $H(X \mid Y = y)$ may be greater than $H(X)$ for some y — "contradicting evidence".

Chain Rule of Mutual Information

The **conditional mutual information** of variables X and Y given Z is defined as

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For random variables Y and $X_1, \ldots, X_n$ we have

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Independence among $X_1, \ldots, X_n$ implies

$$I(Y\,;\,X_1, \ldots, X_n) = \sum_{i=1}^{n} I(Y\,;\,X_i)\ .$$

Data Processing Inequality

Let X, Y, Z be (discrete) random variables. If Z is *conditionally independent of* X *given* Y, i.e., if we have

$$p_{Z|X,Y}(z \mid x, y) = p_{Z|Y}(z \mid y) \quad \text{for all } x, y, z,$$

then X, Y, Z form a **Markov chain** $X \to Y \to Z$.

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For instance, Y is a "noisy" measurement of X, and $Z = f(Y)$ is the outcome of deterministic data processing performed on Y, then we have $X \to Y \to Z$.

This implies that

$$I(X\, ;\, Z \mid Y) = H(X \mid Y) - H(X \mid Y, Z) = 0 \ .$$

When Y is known, Z doesn't give any extra information about X. (and vice versa).

Data Processing Inequality

Assuming that $X \to Y \to Z$ is a Markov chain, we get

$$\begin{aligned} I(X\,;\,Y,Z) &= I(X\,;\,Z) + I(X\,;\,Y \mid Z) \\ &= I(X\,;\,Y) + I(X\,;\,Z \mid Y)\ . \end{aligned}$$

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Now, because $I(X\,;\,Z \mid Y) = 0$, and $I(X\,;\,Y \mid Z) \geq 0$, we obtain:

Data Processing Inequality

If $X \to Y \to Z$ is a Markov chain, then we have

$$I(X\,;\,Z) \leq I(X\,;\,Y)\ .$$

No data-processing can increase the amount of information that we have about X.

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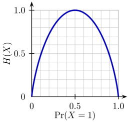

AEP

If $X_1, X_2, \ldots$ is a sequence of independent and identically distributed (i.i.d.) r.v.'s with domain $\mathcal{X}$ and pmf p_X, then

$$\log_2 \frac{1}{p_X(X_1)}, \log_2 \frac{1}{p_X(X_2)}, \ldots$$

is also an i.i.d. sequence of r.v.'s.

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is also an i.i.d. sequence of r.v.'s.

By the definition of entropy and the i.i.d. assumption, the expected values of the elements of the above sequence are all equal to the entropy:

$$E\left[\log_2 \frac{1}{p_X(X_i)}\right] = \sum_{x \in \mathcal{X}} p_X(x) \log_2 \frac{1}{p_X(x)} = H(X) \quad \text{for all } i \in \mathbb{N}.$$

AEP

By the (weak) law of large numbers, the average

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\log_2 p_X(X_i)}$$

converges in probability to its mean, i.e., the entropy:

$$\lim_{n\to\infty}\Pr\left[\left|\frac{1}{n}\sum_{i=1}^{n}\log_2\frac{1}{p_X(X_i)}-H(X)\right|<\epsilon\right]=1 \quad \text{for all } \epsilon>0.$$

AEP

The i.i.d. assumption is equivalent to

$$p_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^{n} p_X(x_i) \quad .$$

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The i.i.d. assumption is equivalent to

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Asymptotic Equipartition Property (AEP)

For i.i.d. sequences, we have

$$\lim_{n\to\infty} \Pr\left[\left|\frac{1}{n}\log_2 \frac{1}{p_{X_1,\ldots,X_n}(x_1,\ldots,x_n)} - H(X)\right| < \epsilon\right] = 1$$

for all $\epsilon > 0$.

AEP

The AEP states that for any $\epsilon > 0$, and large enough n, we have

$$\Pr\left[\left|\frac{1}{n}\log_2 \frac{1}{p_{X_1,\ldots,X_n}(x_1,\ldots,x_n)} - H(X)\right| < \epsilon\right] \approx 1 \ .$$

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"Almost all events are almost equally likely."

Typical Sets

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The **typical set** $A_\epsilon^{(n)}$ is the set of sequences $(x_1, \ldots, x_n) \in \mathcal{X}^n$ with the property:

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The AEP states that

$$\lim_{n\to\infty} \Pr\left[X^n \in A_\epsilon^{(n)}\right] = 1 \ .$$

In particular, for any $\epsilon > 0$, and large enough n, we have

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The number of all possible sequences $(x_1, \ldots, x_n) \in \mathcal{X}^n$ of length n is $|\mathcal{X}|^n$.

The maximum of entropy is $\log_2 |\mathcal{X}|$. If $H(X) = \log_2 |\mathcal{X}|$, we obtain

$$\left| A_\epsilon^{(n)} \right| \approx 2^{nH(X)} = 2^{n \log_2 |\mathcal{X}|} = |\mathcal{X}|^n \ ,$$

i.e., the typical set can be as large as the whole set $\mathcal{X}^n$.

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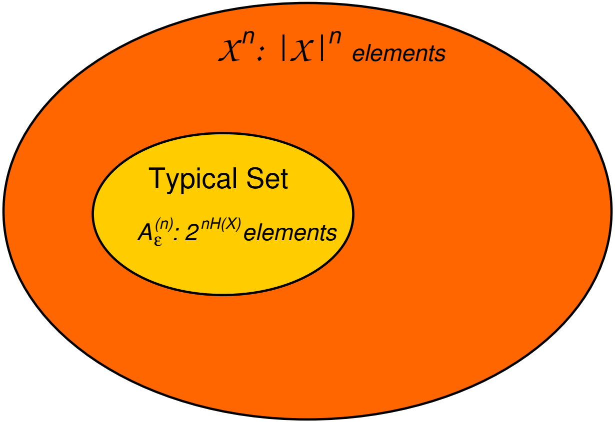
The number of all possible sequences $(x_1, \ldots, x_n) \in \mathcal{X}^n$ of length n is $|\mathcal{X}|^n$.

However, for $H(X) < \log_2 |\mathcal{X}|$, the number of sequences in $A_\epsilon^{(n)}$ is *exponentially smaller* than $|\mathcal{X}|^n$:

$$\frac{2^{nH(X)}}{2^{n \log_2 |\mathcal{X}|}} = 2^{-n\delta} \underset{n\to\infty}{\longrightarrow} 0 \ , \quad \text{if } \delta = \log_2 |\mathcal{X}| - H(X) > 0.$$

Typical Sets



A (relatively) small set that contains most of the probability mass.

Examples

If the source consists of i.i.d. bits $\mathcal{X} = \{0, 1\}$ with $p = p_X(1) = 1 - p_X(0)$, then we have

$$p_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^{n} p_X(x_i) = p^{\sum x_i}(1-p)^{n-\sum x_i} \quad ,$$

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where $\sum x_i$ is the number of 1's in x^n.

In this case, the typical set $A_\epsilon^{(n)}$ consists of sequences for which $\sum x_i$ is close to np. For such strings, we have

$$\log_2 \frac{1}{p(x_1, \ldots, x_n)} \approx \log_2 \frac{1}{p^{np}(1-p)^{n(1-p)}}$$
$$\approx n\left(p \log_2 \frac{1}{p} + (1-p)\log_2 \frac{1}{1-p}\right) = H(X) \quad .$$

Examples

If the source consists of i.i.d. rolls of a die $\mathcal{X} = \{1, 2, 3, 4, 5, 6\}$ with $p_j = p_X(j)$, $j \in \mathcal{X}$, then we have

$$p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_X(x_i) = \prod_{j=1}^{6} p_j^{k_j} \ ,$$

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The AEP Code

We now construct a code from source strings $(x_1, \ldots, x_n) \in \mathcal{X}^n$ to binary sequences $\{0, 1\}^*$ of arbitrary length.

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The code we will construct has expected per-symbol codeword length arbitrarily close to the entropy

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! This is **the best achievable rate** for uniquely decodable codes.

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Since the decoder must be able to tell which kind of a string it is decoding, we prefix the code by a 0 if $x^n \in A_\epsilon^{(n)}$ or by 1 if not. This adds one more bit in either case.

Expected Codelength of the AEP Code

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&+ (n \log_2 |\mathcal{X}| + 2) \Pr\left[X^n \notin A_\epsilon^{(n)}\right] \\
&\leq n(H(X)+\epsilon) + n \log |\mathcal{X}| \epsilon + 2 \quad \text{(AEP)} \\
&= n(H(X)+\epsilon') \ ,
\end{aligned}$$

where $\epsilon' = \epsilon + \epsilon \log_2 |\mathcal{X}| + \frac{2}{n}$ can be made arbitrarily small by choosing $\epsilon > 0$ small enough, and letting n become large enough.

Optimality of the AEP Code

Dividing this bound by n gives the expected per-symbol codelength of the “AEP code”:

$$E\left[\frac{1}{n}\ell(X^n)\right] \leq H(X) + \epsilon$$

for any $\epsilon > 0$ and n large enough.

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for any $\epsilon > 0$ and n large enough.

Optimality: By AEP, there are about $2^{nH(X)}$ sequences that have probability about $2^{-nH(X)}$. We can assign a codeword shorter than $n(H(X) - \delta)$ to only a proportion of less than $2^{-n\delta}$ of these sequences (by a counting argument), and hence the expected per-symbol codeword length must be about $H(X)$ or more.

Noiseless Source Coding Theorem

These two statements give the

9. THE FUNDAMENTAL THEOREM FOR A NOISELESS CHANNEL

We will now justify our interpretation of H as the rate of generating information by proving that H determines the channel capacity required with most efficient coding.

Theorem 9: Let a source have entropy H (bits per symbol) and a channel have a capacity C (bits per second). Then it is possible to encode the output of the source in such a way as to transmit at the average rate $\frac{C}{H} - \epsilon$ symbols per second over the channel where ϵ is arbitrarily small. It is not possible to transmit at an average rate greater than $\frac{C}{H}$.

(Shannon, 1948)

In the noiseless setting with binary code alphabet, the channel capacity is $C = \log_2 |\{0, 1\}| = 1$.

The theorem says that the achievable rates are given by

$$R = \lim_{n\to\infty} \frac{n}{\ell(x^n)} < \frac{1}{H(X)} .$$

Next Week

For next week, do the first assignment if not yet done.

Next lecture will be about *practical* codes: Shannon–Fano, Huffman, arithmetic code, (Lempel–Ziv, ... ?)

We will also form the project groups. **Participation mandatory.**