Three Concepts: Information

Lecture 4: Source Coding: Practice

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Concentric Circular Tower
(David Huffman)

"Design with the help of binary code (0 and 1) the most efficient method to represent characters, figures and symbols."

(Assignment at Prof. R.M. Fano’s 1952 MIT Information Theory course.)
1 Codes

- Decodable Codes
- Prefix Codes
- Kraft-McMillan Theorem
1. Codes
   - Decodable Codes
   - Prefix Codes
   - Kraft-McMillan Theorem

2. Optimal Codes
   - Entropy Lower Bound
   - Shannon-Fano
   - Huffman
1. Codes
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2. Optimal Codes
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   - Shannon-Fano
   - Huffman

3. Below Entropy
   - Problems with Symbol Codes
   - Two-Part Codes
   - Block Codes
Extension Code

A (binary) symbol code $C : \mathcal{X} \rightarrow \{0, 1\}^*$ is a mapping from the alphabet $\mathcal{X}$ to the set of finite binary sequences.
A (binary) **symbol code** $C : \mathcal{X} \to \{0,1\}^*$ is a mapping from the alphabet $\mathcal{X}$ to the set of finite binary sequences.

The **extension** of code $C$ is the mapping $C^* : \mathcal{X}^* \to \{0,1\}^*$ obtained by concatenating the codewords $C(x_i)$ for each input symbol $x_i$:

$$C^*(x_1, x_2, \ldots, x_n) = C(x_1)C(x_2)\ldots C(x_n).$$

![Diagram of extension code]

**Extension Code**

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\[
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\]

\[
\text{INPUT STRING} \ldots
\]

\[
100100011100110101111111010011\ldots
\]
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Decodable Codes

Decodable Code

Code $C$ is (uniquely) **decodable** iff its extension $C^*$ is a one-to-one mapping, i.e., iff

$$(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) \Rightarrow C^*(x_1, \ldots, x_n) \neq C^*(y_1, \ldots, y_n).$$
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X A code with codewords $\{0, 1, 10, 11\}$ is not uniquely decodable: What does 10 mean?
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**X** A code with codewords $\{0, 1, 10, 11\}$ is *not* uniquely decodable: What does 10 mean?

**✓** A code with codewords $\{00, 01, 10, 11\}$ *is* uniquely decodable: Each pair of bits can be decoded individually.

**✓** A code with codewords $\{0, 01, 011, 0111\}$ is also uniquely decodable: What does 0011 mean?
Prefix Codes

An important subset of decodable codes is the set of prefix-free codes.

Prefix Code

A code \( C : \mathcal{X} \to \{0, 1\}^* \) is called a prefix code iff no codeword is a prefix of another.

It is easily seen that all prefix codes are uniquely decodable: each symbol can be decoded as soon as its codeword is read. Therefore, prefix codes are also called instantaneous codes.
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An important subset of decodable codes is the set of **prefix(-free) codes**.

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It is easily seen that all prefix codes are uniquely decodable: each symbol can be decoded as soon as its codeword is read. Therefore, prefix codes are also called *instantaneous* codes.

- **X** A code with codewords \( \{0, 01, 011, 0111\} \) is uniquely decodable *but not prefix-free*: e.g., 0 is a prefix of 01.
- **✓** A code with codewords \( \{0, 10, 110, 111\} \) *is* prefix-free.
Kraft Inequality

The codeword lengths of a prefix codes satisfy the following important property.

Kraft Inequality

The codeword lengths $\ell_1, \ldots, \ell_m$ of any (binary) prefix code satisfy

$$\sum_{i=1}^{m} 2^{-\ell_i} \leq 1 .$$

Conversely, given a set of codeword lengths that satisfy this inequality, there is a prefix code with these codeword lengths.
## Kraft Inequality

### Total budget

<table>
<thead>
<tr>
<th>Codewords</th>
<th>000</th>
<th>001</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>000</td>
<td>001</td>
</tr>
<tr>
<td>00</td>
<td>000</td>
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<td>010</td>
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<td>101</td>
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<td>10</td>
<td>110</td>
<td>111</td>
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<tr>
<td>11</td>
<td>110</td>
<td>111</td>
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</tbody>
</table>

\[ \sqrt{\text{Total budget}} \]

### Codewords

\{0, 10, 110, 111\}

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### Kraft Inequality

<table>
<thead>
<tr>
<th>Total budget</th>
<th>00</th>
<th>01</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>000</td>
<td>010</td>
<td>100</td>
<td>110</td>
</tr>
<tr>
<td>01</td>
<td>001</td>
<td>011</td>
<td>101</td>
<td>111</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>101</td>
<td>110</td>
<td>111</td>
</tr>
</tbody>
</table>

Kraft inequality violated. ⇒ Not decodable.
# Kraft Inequality

<table>
<thead>
<tr>
<th>Total budget</th>
<th>00</th>
<th>01</th>
<th>10</th>
<th>11</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td><strong>00</strong></td>
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<td><strong>0110</strong></td>
<td><strong>1110</strong></td>
<td><strong>1111</strong></td>
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</tbody>
</table>

- Total budget:
  - 0: 00, 001, 010, 011
  - 01: 010, 0110, 0111
  - 10: 100, 1001
  - 11: 110, 1101

- **Fixed-length code**

---

The Kraft inequality states that for a set of decodable codes, the sum of the probabilities of the source symbols must be less than or equal to 1. This is illustrated in the table above, where each code is prefixed with a '0' or '1' to ensure uniqueness and decodability. The table shows that the total budget is maintained within the Kraft inequality constraints.
Kraft Inequality

Decodable & prefix-free

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Total budget

Kraft?
Decodable?
Prefix-free?

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Kraft Inequality

<table>
<thead>
<tr>
<th>Total budget</th>
<th>Kraft?</th>
<th>Decodable?</th>
<th>Prefix-free?</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
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<tr>
<td>1</td>
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</tbody>
</table>

Kraft-McMillan Theorem

- Kraft Inequality
- Decodable Codes
- Prefix Codes
- Optimal Codes
- Below Entropy

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### Kraft Inequality

<table>
<thead>
<tr>
<th>Total budget</th>
<th>Kraft?</th>
<th>Decodable?</th>
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<tbody>
<tr>
<td>00</td>
<td>0</td>
<td>000</td>
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<td>001</td>
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<td>0100, 0101</td>
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<td>111</td>
<td>7</td>
<td>111</td>
<td>1101, 1110, 1111</td>
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**Kraft Inequality**

- **Kraft?** (valid if the sum of the binary logarithms of the probabilities is less than or equal to the total budget)
- **Decodable?** (true if the code is Kraft)
- **Prefix-free?** (true if the code is Kraft and the code is prefix-free)

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Kraft? ✓ Decodable? ✗ Prefix-free? ✗
Question: What if the inequality is satisfied strictly, i.e., the sum of the terms in the sum equals less than one:

$$\sum_{i=1}^{m} 2^{-\ell_i} < 1.$$
Kraft Inequality

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\[
\sum_{i=1}^{m} 2^{-\ell_i} < 1.
\]

Then it is possible to make the codewords shorter and still have a decodable (prefix) code.
Kraft Inequality

Not all of budget used. ⇒ Some codewords can be made shorter.
Kraft Inequality

"Kraft tight" / complete code.
The Kraft inequality restricts the codeword lengths of prefix codes. Could we do much better if we would only require decodability?
Kraft–McMillan Theorem

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In fact it can be shown that we do not lose anything at all!
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In fact it can be shown that we do not lose anything at all!

Kraft–McMillan Theorem

The codeword lengths \( \ell_1, \ldots, \ell_m \) of any uniquely decodable (binary) code satisfy

\[
\sum_{i=1}^{m} 2^{-\ell_i} \leq 1.
\]

Conversely, given a set of codeword lengths that satisfy this inequality, there is a uniquely decodable (prefix) code with these codeword lengths.
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   - Decodable Codes
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2. Optimal Codes
   - Entropy Lower Bound
   - Shannon-Fano
   - Huffman

3. Below Entropy
   - Problems with Symbol Codes
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   - Block Codes
Let $\ell_1, \ldots, \ell_m$ be the codeword lengths of a uniquely decodable code $C : \mathcal{X} \rightarrow \{0,1\}^*$. By the Kraft-McMillan theorem we have

$$c = \sum_{i=1}^{m} 2^{-\ell_i} \leq 1.$$
Let $\ell_1, \ldots, \ell_m$ be the codeword lengths of a uniquely decodable code $C : \mathcal{X} \rightarrow \{0, 1\}^*$. By the Kraft-McMillan theorem we have

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Define a probability mass function $p : \mathcal{X} \rightarrow [0, 1]$ as follows:

$$p_i = \frac{2^{-\ell_i}}{c}$$

where $c$ is given above.
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Define a probability mass function $p : \mathcal{X} \rightarrow [0, 1]$ as follows:

$$p_i = \frac{2^{-\ell_i}}{c} \iff \ell_i = \log_2 \frac{c}{p_i},$$

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Function $p$ is indeed a pmf:

1. **Non-negative:** $p(x) \geq 0$ for all $x \in \mathcal{X}$. 
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Function $p$ is indeed a pmf:

1. Non-negative: $p(x) \geq 0$ for all $x \in \mathcal{X}$.
2. Sums to one: $\sum_{x \in \mathcal{X}} p(x) = \sum_{i=1}^{m} \frac{1}{c} 2^{-\ell_i} = \frac{c}{c} = 1$.
Assuming that the code is “Kraft tight”, \( c = 1 \), then under the pmf \( p \) corresponding to the codeword lengths \( \ell_1, \ldots, \ell_m \), the expected codeword length is

\[
E[\ell(X)] = \sum_{i=1}^{m} 2^{-\ell_i} \ell_i
\]
Code lengths and Probabilities

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$$E[\ell(X)] = \sum_{i=1}^{m} 2^{-\ell_i} \ell_i$$

$$= \sum_{i=1}^{m} p_i \log_2 \frac{1}{p_i} = H(X) .$$
Codelengths and Probabilities

Assuming that the code is “Kraft tight”, \( c = 1 \), then under the pmf \( p \) corresponding to the codeword lengths \( \ell_1, \ldots, \ell_m \), the expected codeword length is

\[
E[\ell(X)] = \sum_{i=1}^{m} 2^{-\ell_i} \ell_i \\
= \sum_{i=1}^{m} p_i \log_2 \frac{1}{p_i} = H(X) .
\]

This is the best we can hope for:

The expected codelength of any uniquely decodable code is at least the entropy:

\[
E[\ell(X)] \geq H(X) .
\]
Entropy Lower Bound

\[ E[\ell(X)] \geq H(X) \]
Entropy Lower Bound

\[ E[\ell(X)] \geq H(X). \]

**Proof.**

\[
E[\ell(X)] - H(X) = \sum_{x \in X} p(x) \ell(x) - \sum_{x \in X} p(x) \log_2 \frac{1}{p(x)}
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Entropy Lower Bound

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Proof.

\[
E[\ell(X)] - H(X) = \sum_{x \in \mathcal{X}} p(x) \ell(x) - \sum_{x \in \mathcal{X}} p(x) \log_2 \frac{1}{p(x)}
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\[
= \sum_{x \in \mathcal{X}} p(x) \log_2 \frac{1}{2^{-\ell_x}} - \sum_{x \in \mathcal{X}} p(x) \log_2 \frac{1}{p(x)}
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\]

\[
= \sum_{x \in X} p(x) \left[ \log_2 \frac{p(x)}{q(x)} + \log_2 \frac{1}{c} \right]
\]

\[ q(x) = \frac{2^{-\ell(x)}}{c} \]
Entropy Lower Bound

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E[\ell(X)] - H(X) = \sum_{x \in \mathcal{X}} p(x) \ell(x) - \sum_{x \in \mathcal{X}} p(x) \log_2 \frac{1}{p(x)} \\
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= \sum_{x \in \mathcal{X}} p(x) \log_2 \frac{p(x)}{2^{-\ell_x}} \\
= \sum_{x \in \mathcal{X}} p(x) \left[ \log_2 \frac{p(x)}{q(x)} + \log_2 \frac{1}{c} \right] \\
= D(p \parallel q) + \log_2 \frac{1}{c} \geq 0.
\]

\[ q(x) = 2^{-\ell(x)} \]

Teemu Roos  
Three Concepts: Information
Entropy Lower Bound

So what have we learned?

For decodable symbols codes:

1. $E[\ell(X)] - H(X) = D(p∥q) + \log_2 \frac{1}{c}$, where $q(x) = 2^{-\ell(x)} c$.

2. $E[\ell(X)] \geq H(X)$.

3. If $\ell(x) = \log_2 \frac{1}{p(x)}$, then $E[\ell(X)] = H(X)$.

Optimal!

Note also that for a sequence $X_1, \ldots, X_n$ the expected codelength becomes $E[\ell(X_1, \ldots, X_n)] = \sum_{i=1}^{n} E[\ell(X_i)] = n H(X)$.

By Shannon's Noiseless Channel Coding Theorem, this is optimal among all codes, not only symbol codes.

Fine print: only if $X_i$ i.i.d.
So what have we learned? For decodable symbols codes:

1. \( E[\ell(X)] - H(X) = D(p \parallel q) + \log_2 \frac{1}{c} \), where \( q(x) = \frac{2^{-\ell(x)}}{c} \).
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Entropy Lower Bound

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Entropy Lower Bound

So what have we learned? For decodable symbols codes:

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Note also that for a sequence \( X_1, \ldots, X_n \) the expected codelength becomes

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E[\ell(X_1, \ldots, X_n)] = E \left[ \sum_{i=1}^{n} \ell(X_i) \right] = \sum_{i=1}^{n} E[\ell(X_i)] = nH(X) .
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By Shannon’s Noiseless Channel Coding Theorem, this is optimal among all codes, not only symbol codes. Fine print: only if \( X_i \) i.i.d.!
The only problem with the \( \ell(x) = \log_2 \frac{1}{p(x)} \) codeword choice is the requirement that codeword lengths must be integers (try to think about a codeword with length 0.123, for instance), while the so obtained \( \ell \) is not in general an integer.
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The simplest solution is to round upwards:

**Shannon-Fano Code**

Given a pmf, the **Shannon-Fano code** has the codeword lengths

$$\ell(x) = \left\lceil \log_2 \frac{1}{p(x)} \right\rceil$$

for all $x \in \mathcal{X}$. 

Teemu Roos

Three Concepts: Information
## Shannon-Fano: Example

$$H(X) = 4.03$$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$p(X)$</th>
<th>$\log_2 \frac{1}{p(X)}$</th>
<th>$\ell(X)$</th>
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<tbody>
<tr>
<td>a</td>
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### Shannon-Fano: Example

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$H(X) = 4.03$

Shannon-Fano:
### Shannon-Fano: Example

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\[ H(X) = 4.03 \]

**Shannon-Fano:**

1. Sort by probability.
Shannon-Fano: Example

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\[ H(X) = 4.03 \]

Shannon-Fano:

1. Sort by probability.
2. Choose codewords in order, avoiding prefixes. ("Kraft table"!)

Teemu Roos
Three Concepts: Information
### Shannon-Fano: Example

<table>
<thead>
<tr>
<th>Total budget</th>
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</table>

Codeword lengths (3, 4, 4, 4, 5, 5, 5, 5, ... 10, 10, 11)
### Shannon-Fano: Example

<table>
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Shannon-Fano: Example

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<th>$C(X)$</th>
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<tbody>
<tr>
<td>x</td>
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<td>2.2</td>
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$H(X) = 4.03$.

$E[\ell(X)] = 4.60$.

$E[\ell(X)] - H(X) = 0.57$.  

Teemu Roos

Three Concepts: Information
### Shannon-Fano: Example

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The expected codeword length of the Shannon-Fano code is

\[ E[\ell(X)] = E \left[ \left\lceil \log_2 \frac{1}{p(X)} \right\rceil \right] \leq E \left[ \log_2 \frac{1}{p(X)} + 1 \right] = H(X) + 1. \]
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In the Alice example we had

\[
E[\ell(X)] - H(X) = 4.60 - 4.03 = 0.57 \leq 1.
\]
Shannon-Fano Code

Consider the Shannon-Fano code for Alice in Wonderland. The longest codewords are as follows:

<table>
<thead>
<tr>
<th>X</th>
<th>p(X)</th>
<th>log₂ ( \frac{1}{p(X)} )</th>
<th>ℓ(X)</th>
<th>C(X)</th>
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<tbody>
<tr>
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<td>0.0108</td>
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<td>k</td>
<td>0.0083</td>
<td>6.8</td>
<td>7</td>
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<tr>
<td>v</td>
<td>0.0061</td>
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<td>q</td>
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Can you find a way to improve the code without violating the prefix-free property?
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Huffman Code

So the Shannon-Fano code is not the optimal symbol code. This is where Professor Fano and a student called David Huffman enter:
Huffman Code

So the Shannon-Fano code is not the optimal symbol code. This is where Professor Fano and a student called David Huffman enter:

"Design with the help of binary code (0 and 1) the most efficient method to represent characters, figures and symbols."
David Huffman (1925–1999)
Huffman Code: Algorithm

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1. Sort all symbols by their probabilities \( p_i \).
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See the demo at [www.cs.auckland.ac.nz/software/AlgAnim/huffman.html](http://www.cs.auckland.ac.nz/software/AlgAnim/huffman.html)
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The reason why the Huffman code is the optimal symbol code (shortest expected code length) is roughly as follows:

1. If \( p(x) > p(y) \), then \( \ell(x) \leq \ell(y) \).
2. The longest two codewords have the same length.
3. The longest two codewords differ only at the last bit and correspond to the two least probable symbols.

Points 2 & 3 suggest the first step of Huffman's algorithm. Any subtree must satisfy the same conditions $\Rightarrow$ Induction.

Note that since Shannon-Fano gives $E[\ell(X)] \leq H(X) + 1$, and Huffman is optimal, Huffman must satisfy the same bound.
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1 Codes
   - Decodable Codes
   - Prefix Codes
   - Kraft-McMillan Theorem

2 Optimal Codes
   - Entropy Lower Bound
   - Shannon-Fano
   - Huffman

3 Below Entropy
   - Problems with Symbol Codes
   - Two-Part Codes
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Now we have found the optimal symbols code with expected codelength $E[\ell(X)] \leq H(X) + 1$. Are we done?

No. (At least) three problems remain:

1. The one extra bit, $H(X) + 1$. Can make all the difference if $H(X)$ is small.
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Write the distribution (or code) in the beginning of the file.
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Usually the overhead is minor compared to the total file size.
Block Codes

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Combine successive symbols into blocks and treat blocks as symbols. ⇒ One extra bit per block.
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Allows modeling of dependence.
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Combining solutions to problems 1–3, we get **two-part block codes**: Write first the joint distribution of blocks of $N$ symbols, and then encode using blocks of length $N$. 
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The size of the first part (distribution/code) grows with $N$, but the performance of the block code get better.
Block Codes

Combining solutions to problems 1–3, we get **two-part block codes**: Write first the joint distribution of blocks of $N$ symbols, and then encode using blocks of length $N$.

The size of the first part (distribution/code) grows with $N$, but the performance of the block code get better.

**Complexity Tradeoff**

Find suitable balance between complexity of the model (increases with $N$) and codelength of data given model (decreases with $N$).

$\Rightarrow$ **Minimum Description Length (MDL) Principle**
Adaptive Codes

Alternative Solution to Problems 2 & 3:

Adaptive Codes

For each symbol (or a block of symbols), we can construct a code based on the probability $p(x_{\text{new}} | x_1, \ldots, x_n)$. 

This may lead to computational problems since the code tree has to be constantly updated.

Block coding with long blocks is another solution, but it introduces delay in decoding: the first symbol can be read only after the whole block is decoded.

Arithmetic coding avoids "all problems": adaptive, spreads the one additional bit over the whole sequence, and can be decoded instantaneously.
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**Arithmetic coding** avoids “all problems”: adaptive, spreads the one additional bit over the whole sequence, and can be decoded instantaneously. ⇒ Read the material.