Three Concepts: Information

Lecture 4: Source Coding: Practice

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Fall 2007

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Concentric Circular Tower
(David Huffman)

[Photo: Tony Grant. Courtesy of the Huffman family.]

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"Design with the help
of binary code (0 and
1) the most efficient
method to represent
characters, figures
and symbols."
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(Assignment at Prof. R.M. Fano's 1952 MIT Information Theory course.)

1. Codes
   - Decodable Codes
   - Prefix Codes
   - Kraft-McMillan Theorem

Extension Code

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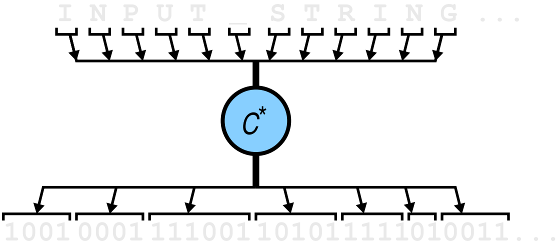

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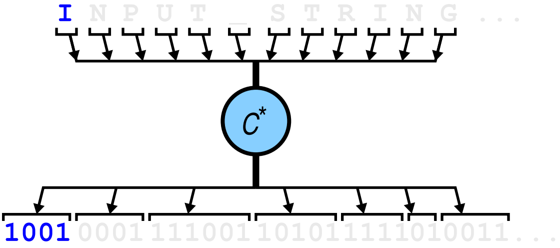

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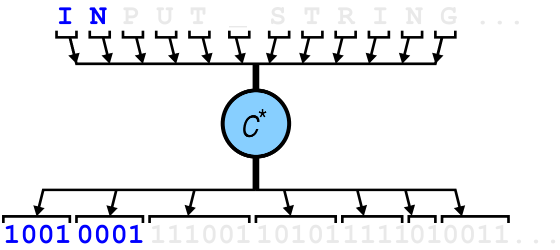

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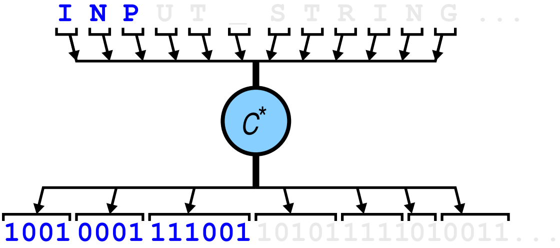

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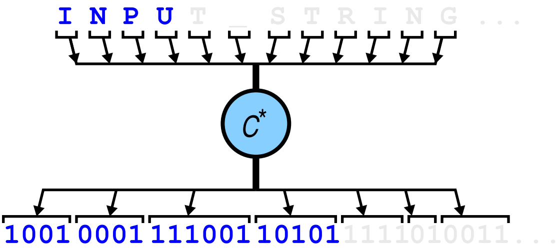

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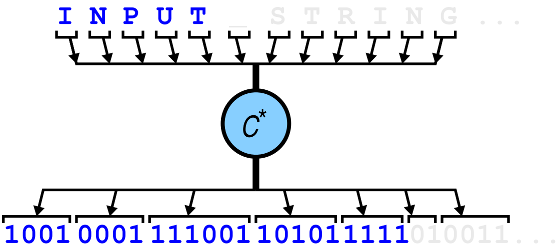

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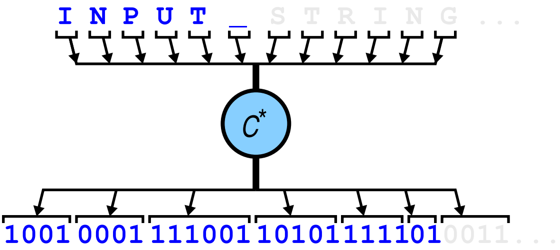

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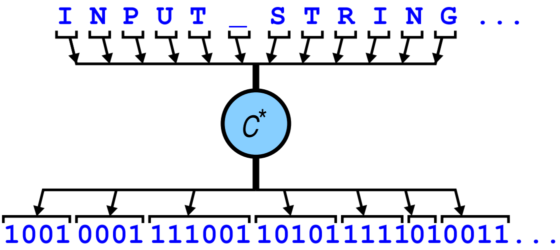

Decodable Codes

Decodable Code

Code C is (uniquely) **decodable** iff its extension C^* is a one-to-one mapping, i.e., iff

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- ✗ A code with codewords $\{0, 1, 10, 11\}$ is *not* uniquely decodable: What does 10 mean?
- √ A code with codewords $\{00, 01, 10, 11\}$ *is* uniquely decodable: Each pair of bits can be decoded individually.
- √ A code with codewords $\{0, 01, 011, 0111\}$ is also uniquely decodable: What does 0011 mean?

Prefix Codes

An important subset of decodable codes is the set of **prefix(-free) codes**.

Prefix Code

A code $C : \mathcal{X} \to \{0,1\}^*$ is called a **prefix code** iff no codeword is a prefix of another.

It is easily seen that all prefix codes are uniquely decodable: each symbol can be decoded as soon as its codeword is read. Therefore, prefix codes are also called *instantaneous* codes.

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- ✗ A code with codewords $\{0, 01, 011, 0111\}$ is uniquely decodable *but not prefix-free*: e.g., 0 is a prefix of 01.
- ✓ A code with codewords $\{0, 10, 110, 111\}$ *is* prefix-free.

Kraft Inequality

The codeword lengths of a prefix codes satisfy the following important property.

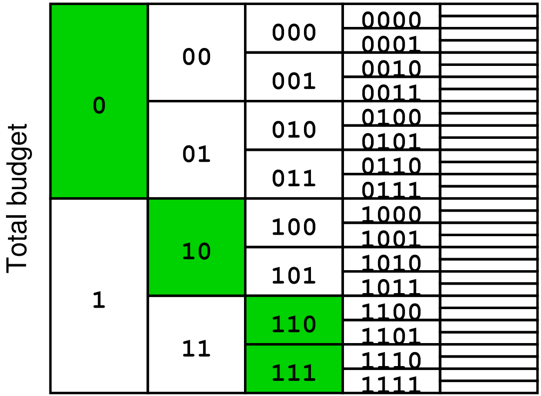
Kraft Inequality

The codeword lengths $\ell_1, \ldots, \ell_m$ of any (binary) prefix code satisfy

$$\sum_{i=1}^{m} 2^{-\ell_i} \leq 1 \ .$$

Conversely, given a set of codeword lengths that satisfy this inequality, there is a prefix code with these codeword lengths.

Kraft Inequality



√ Codewords $\{0, 10, 110, 111\}$

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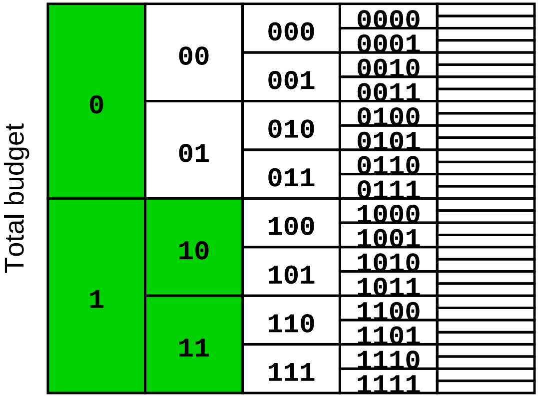
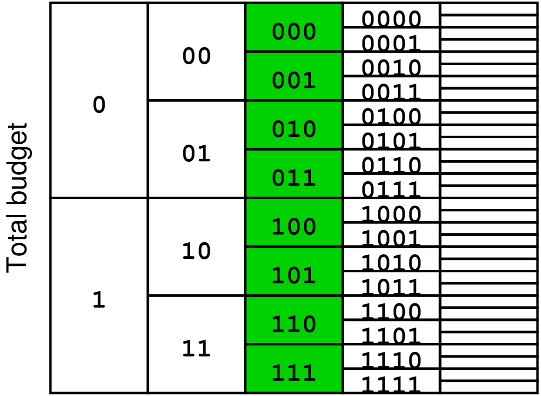


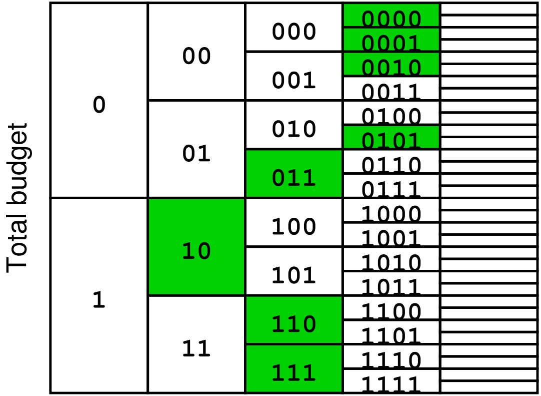
✗ Kraft inequality violated. $\Rightarrow$ Not decodable.

Kraft Inequality



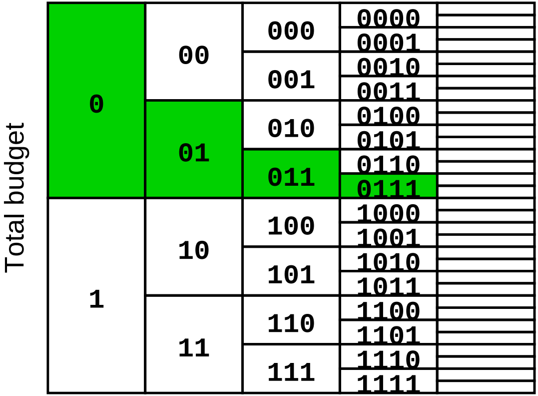
√ Fixed-length code

Kraft Inequality



√ Decodable & prefix-free

Kraft Inequality



Kraft? Decodable? Prefix-free?

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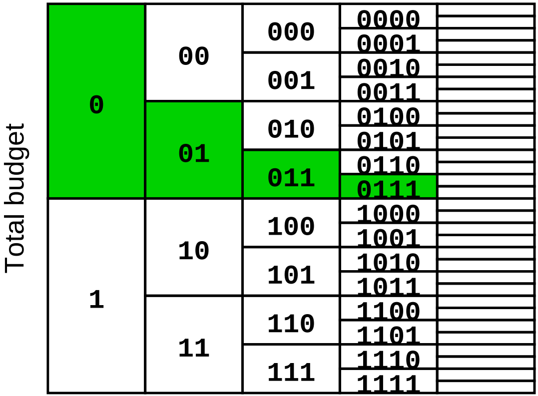

Kraft? √ Decodable? √ Prefix-free? X

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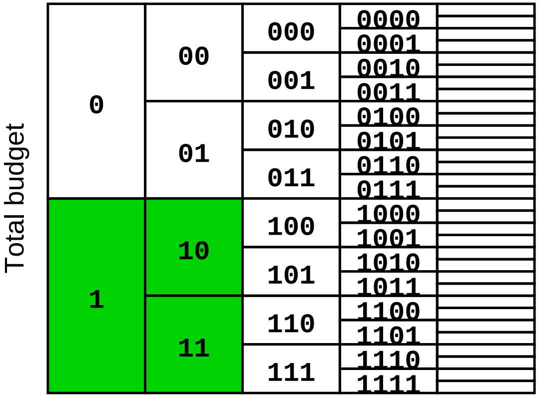

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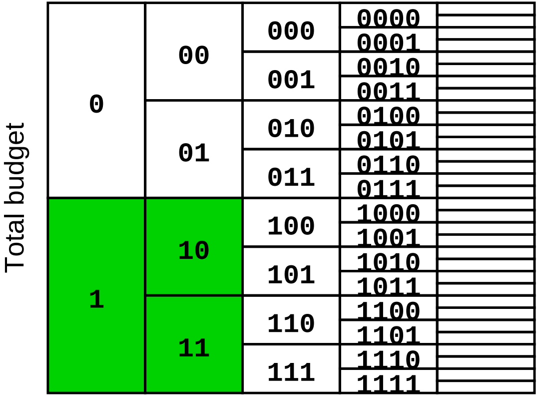

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Question: What if the inequality is satisfied strictly, i.e., the sum of the terms in the sum equals *less* than one:

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Then it is possible to make the codewords shorter and still have a decodable (prefix) code.

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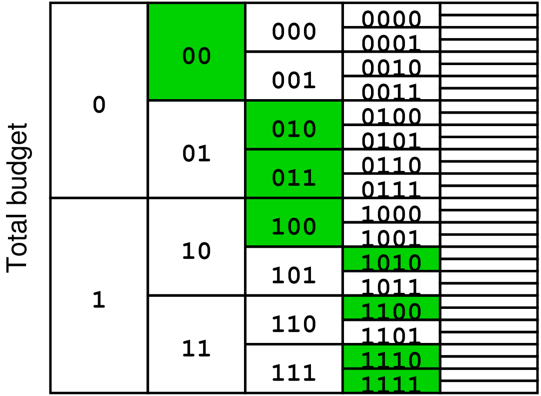
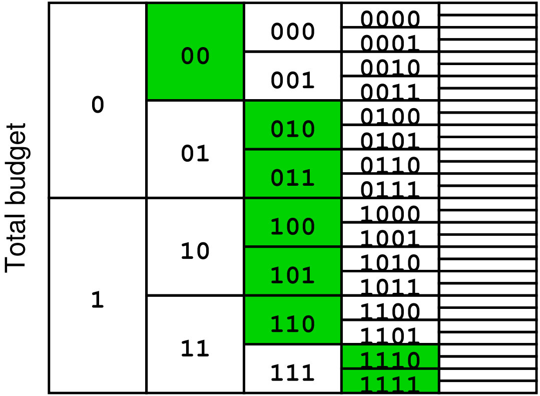


Not all of budget used. $\Rightarrow$ Some codewords can be made shorter.

Kraft Inequality



"Kraft tight" / complete code.

Kraft–McMillan Theorem

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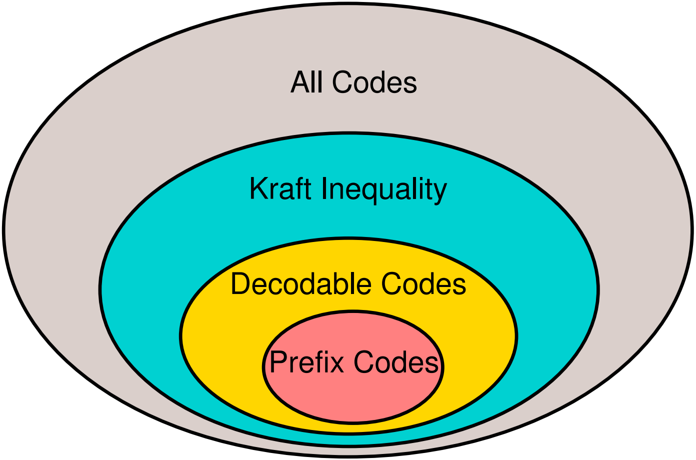
The codeword lengths $\ell_1, \ldots, \ell_m$ of any **uniquely decodable** (binary) code satisfy

$$\sum_{i=1}^{m} 2^{-\ell_i} \leq 1 \ .$$

Conversely, given a set of codeword lengths that satisfy this inequality, there is a uniquely decodable (prefix) code with these codeword lengths.

Kraft-McMillan Theorem & Codes



1. Codes
   - Decodable Codes
   - Prefix Codes
   - Kraft-McMillan Theorem

2. Optimal Codes
   - Entropy Lower Bound
   - Shannon-Fano
   - Huffman

3. Below Entropy
   - Problems with Symbol Codes
   - Two-Part Codes
   - Block Codes

Codelengths and Probabilities

Let $\ell_1, \ldots, \ell_m$ be the codeword lengths of a uniquely decodable code $C : \mathcal{X} \to \{0,1\}^*$. By the Kraft-McMillan theorem we have

$$c = \sum_{i=1}^{m} 2^{-\ell_i} \leq 1 \ .$$

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1. Non-negative: $p(x) \geq 0$ for all $x \in \mathcal{X}$.
2. Sums to one: $\sum_{x \in \mathcal{X}} p(x) = \sum_{i=1}^{m} \frac{1}{c} 2^{-\ell_i} = \frac{c}{c} = 1$.

Codelengths and Probabilities

Assuming that the code is "Kraft tight", $c = 1$, then under the pmf p corresponding to the codeword lengths $\ell_1, \ldots, \ell_m$, the expected codeword length is

$$E[\ell(X)] = \sum_{i=1}^{m} 2^{-\ell_i} \, \ell_i$$

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This is the best we can hope for:

The expected codelength of any uniquely decodable code is at least the entropy:

$$E[\ell(X)] \geq H(X) \ .$$

Entropy Lower Bound

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Proof.

$$E[\ell(X)] - H(X) = \sum_{x \in \mathcal{X}} p(x)\, \ell(x) - \sum_{x \in \mathcal{X}} p(x) \, \log_2 \frac{1}{p(x)}$$

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&= \sum_{x \in \mathcal{X}} p(x) \left[\log_2 \frac{p(x)}{q(x)} + \log_2 \frac{1}{c} \right]
\end{aligned}$$

$$\boxed{q(x) = \frac{2^{-\ell(x)}}{c}}$$

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$$= \sum_{x \in \mathcal{X}} p(x)\left[\log_2 \frac{p(x)}{q(x)} + \log_2 \frac{1}{c}\right] \qquad \boxed{q(x) = \frac{2^{-\ell(x)}}{c}}$$

$$= D(p \parallel q) + \log_2 \frac{1}{c} \geq 0 \quad .$$

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3. If $\ell(x) = \log_2 \frac{1}{p(x)}$, then $E[\ell(X)] = H(X)$. **Optimal!**

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3. If $\ell(x) = \log_2 \frac{1}{p(x)}$, then $E[\ell(X)] = H(X)$. **Optimal!**

Note also that for a sequence $X_1, \ldots, X_n$ the expected codelength becomes

$$E[\ell(X_1, \ldots, X_n)] = E\left[\sum_{i=1}^{n} \ell(X_i)\right]$$

Entropy Lower Bound

So what have we learned? For decodable symbols codes:

1. $E[\ell(X)] - H(X) = D(p \parallel q) + \log_2 \frac{1}{c}$, where $q(x) = \dfrac{2^{-\ell(x)}}{c}$.
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Codelengths and Probabilities

The only problem with the $\ell(x) = \log_2 \frac{1}{p(x)}$ codeword choice is the requirement that codeword lengths must be **integers** (try to think about a codeword with length 0.123, for instance), while the so obtained ℓ is not in general an integer.

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The simplest solution is to round upwards:

Shannon-Fano Code

Given a pmf, the **Shannon-Fano code** has the codeword lengths

$$\ell(x) = \left\lceil \log_2 \frac{1}{p(x)} \right\rceil \quad \text{for all } x \in \mathcal{X}.$$

Alice in Wonderland

Shannon-Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$
a	0.0644	3.9	4
b	0.0108	6.5	7
c	0.0178	5.8	6
d	0.0359	4.7	5
e	0.0991	3.3	4
f	0.0147	6.0	7
g	0.0184	5.7	6
h	0.0535	4.2	5
i	0.0551	4.1	5
j	0.0011	9.8	10
k	0.0083	6.8	7
l	0.0343	4.8	5
	⋮		
y	0.0165	5.9	6
z	0.0005	10.7	11
	0.2111	2.2	3

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h	0.0535	4.2	5
n	0.0516	4.2	5
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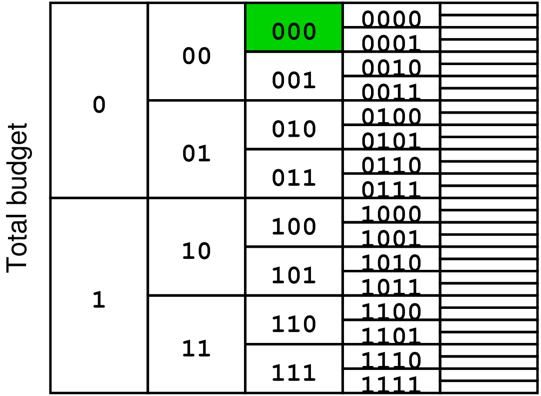
Shannon-Fano:

1. Sort by probability.
2. Choose codewords in order, avoiding prefixes. ("Kraft table"!)

Shannon-Fano: Example



Codeword lengths $(3, 4, 4, 4, 5, 5, 5, 5, \ldots, 10, 10, 11)$

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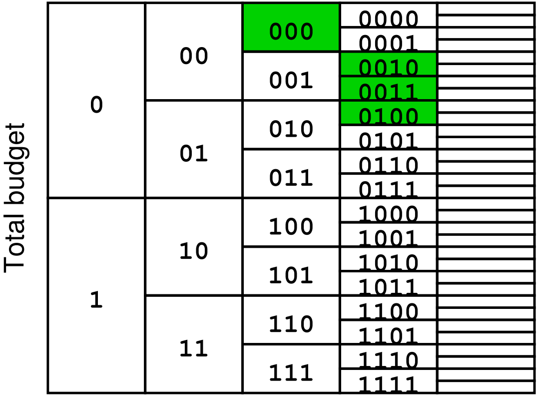

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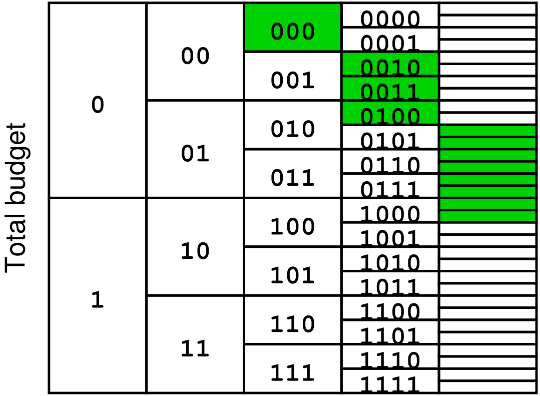

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e	0.0991	3.3	4	0010
t	0.0781	3.6	4	0011
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o	0.0598	4.0	5	01010
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h	0.0535	4.2	5	01100
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$$H(X) = 4.03$$
$$E[\ell(X)] = 4.60$$
$$E[\ell(X)] - H(X) = 0.57$$

Shannon-Fano Code

The expected codeword length of the Shannon-Fano code is

$$E\left[\ell(X)\right] = E\left[\left\lceil \log_2 \frac{1}{p(X)} \right\rceil\right] \leq E\left[\log_2 \frac{1}{p(X)} + 1\right] = H(X) + 1 \ .$$

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In the Alice example we had

$$E[\ell(X)] - H(X) = 4.60 - 4.03 = 0.57 \leq 1 \ .$$

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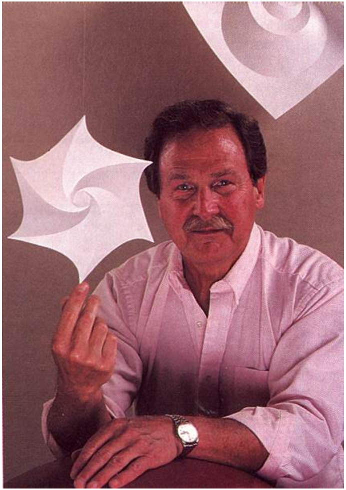
`"Design with the help of binary code (0 and 1) the most efficient method to represent characters, figures and symbols."`

David Huffman (1925–1999)



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See the demo at
`www.cs.auckland.ac.nz/software/AlgAnim/huffman.html`

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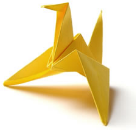
Note that since Shannon-Fano gives $E[\ell(X)] \leq H(X) + 1$, and Huffman is optimal, Huffman must satisfy the same bound.



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Usually the overhead is minor compared to the total file size.

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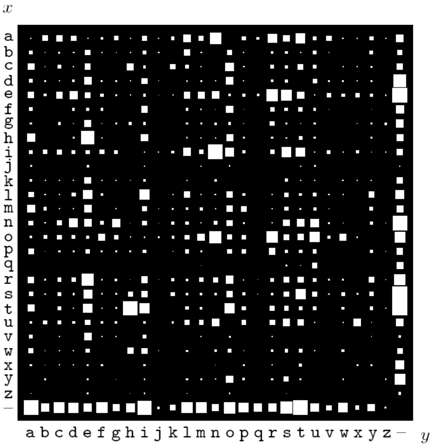
Allows modeling of dependence.

Block Codes


x
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Complexity Tradeoff

Find suitable balance between complexity of the model (increases with N) and codelength of data given model (decreases with N).
$\Rightarrow$ **Minimum Description Length (MDL) Principle**

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Alternative Solution to Problems 2 & 3:

Adaptive Codes

For each symbol (or a block of symbols), we can construct a code based on the probability $p(x_{\text{new}} \mid x_1, \ldots, x_n)$.

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Arithmetic coding avoids "all problems": adaptive, spreads the one additional bit over the whole sequence, and can be decoded instantaneously. $\Rightarrow$ Read the material.