Three Concepts: Information
Lecture 6: MDL Principle (contd.)

Teemu Roos

Complex Systems Computation Group
Department of Computer Science, University of Helsinki

Fall 2007
A.N. Kolmogorov (left) introducing the structure function at a Bernoulli Society meeting, Tallinn, 1973
1. Kolmogorov Complexity
   - Definition
   - Basic Properties
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2. Structure Function
   - Finite Set Models
   - Structure Function
   - Minimal Sufficient Statistic
   - Ideal MDL
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   - Definition
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   - Finite Set Models
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3. MDL Principle
   - Definitions
   - Universal Models
   - Prediction & Model Selection
We probably agree that the string

\[10101010101010101010\ldots10\]

is ‘simple’.

Why?
Kolmogorov Complexity

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(One) Solution: The string can be described briefly:

“10 repeated \(k\) times”.

Remark: ‘Describe’ should be understood as meaning “compute by an algorithm” (a formal procedure that halts).
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Remark: ‘Describe’ should be understood as meaning “compute by an algorithm” (a formal procedure that halts).
Kolmogorov Complexity

Let $U : \{0, 1\}^* \rightarrow \{0, 1\}^* \cup \emptyset$ be a computer that given a (binary) program $p \in \{0, 1\}^*$ either produces a finite (binary) output $U(p) \in \{0, 1\}^*$ or never halts. In the latter case, the output $U(p)$ is said to be undefined ($\emptyset$).
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For a finite string $x \in \{0, 1\}^*$, let $p^*(x)$ be the shortest program for which $U(p^*(x)) = x$.

The **Kolmogorov complexity** of string $x$ is defined as the length of $p^*(x)$:

$$K_U(x) = \min_{p: U(p) = x} |p|.$$
Kolmogorov Complexity

We assume that the set of programs that halt forms a prefix-free set (like symbol codes).
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The advantage of prefix-free programs is that we can concatenate two programs, $p$ and $q$ to form the program $pq$ so that the computer can separate the two programs.
Kolmogorov Complexity

Let $U$ and $V$ be two computers. If computer $U$ is sufficiently ‘rich’, it can emulate computer $V$ so that it outputs the same output as $V$ for any program $p$. 
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Universality

A computer $U$ is said to be universal, if for any other computer $V$ there is a ‘translation’ program $q \in \{0,1\}^*$ (which depends on $V$) such that for all programs $p$ we have

$$U(qp) = V(p),$$

i.e., when given the concatenated program $qp$, computer $U$ outputs the same string as computer $V$ when given the program $p$. 

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Three Concepts: Information
Kolmogorov Complexity

For any \textit{universal} computer \( U \), and any other computer \( V \), we have

\[
K_U(x) \leq K_V(x) + C,
\]

where \( C \) is a constant independent of \( x \).
Kolmogorov Complexity

For any *universal* computer $U$, and any other computer $V$, we have

$$K_U(x) \leq K_V(x) + C ,$$

where $C$ is a constant independent of $x$.

**Proof:** Let $q$ be a the translation program which translates programs of $V$ into programs of $U$, and let $p^*_V(x)$ be the shortest program for which $V(p^*_V(x)) = x$. Then $U(qp^*_V(x)) = x$ so that

$$K_U(x) \leq |qp^*_V(x)| = |p^*_V(x)| + |q| = K_V(x) + |q| .$$
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**Proof:** Let $q$ be a the translation program which translates programs of $V$ into programs of $U$, and let $p_V^*(x)$ be the shortest program for which $V(p_V^*(x)) = x$. Then $U(qp_V^*(x)) = x$ so that

$$K_U(x) \leq |qp_V^*(x)| = |p_V^*(x)| + |q| = K_V(X) + |q| .$$

Based on this property, it can be said that Kolmogorov complexity is the length of the *universally* shortest description of $x$. 
Examples

Examples of (virtually) universal ‘computers’: 

1. C (compiler + operating system + computer),
2. Java (compiler + operating system + computer),
3. Your favorite programming language (compiler/interpreter + OS + computer),
4. Universal Turing machine,
5. Universal recursive function,
6. Lambda calculus,
7. Arithmetics,
8. Game of Life,
9. ...
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Invariance Theorem

From now on we restrict the choice of the computer $U$ in $K_U$ to universal computers.
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**Invariance Theorem**

Kolmogorov complexity is invariant (up to an additive constant) under a change of the universal computer. In other words, for any two universal computers, $U$ and $V$, there is a constant $C$ such that

$$|K_U(x) − K_V(x)| ≤ C \text{ for all } x \in \{0, 1\}^*.$$
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$$|K_U(x) - K_V(x)| \leq C \quad \text{for all } x \in \{0,1\}^* .$$

**Proof:** Since $U$ is universal, we have $K_U(x) \leq K_V(x) + C_1$. Since $V$ is universal, we have $K_V(x) \leq K_U(x) + C_2$. The theorem follows by setting $C = \max\{C_1, C_2\}$. \square
Upper Bound 1

We have the following upper bound on $K_U(x)$:

$$K_U(x) \leq 2|x| + C$$

for some constant $C$ which depends on the computer $U$ but not on the string $x$. 
**Upper Bound 1**

We have the following upper bound on $K_U(x)$:

$$K_U(x) \leq 2|x| + C$$

for some constant $C$ which depends on the computer $U$ but not on the string $x$.

**Proof:** Let $q$ be the program:

- print every even bit that follows until the next odd bit is 0: $x_1 \overline{1} x_2 \overline{1} \ldots x_n 0$.

The length of this program is $2|x| + C$. Prefix-free. \qed
Upper Bound 2

We have the following upper bound on $K_U(x)$:

$$K_U(x) \leq |x| + 2 \log_2 |x| + C$$

for some constant $C$ which depends on the computer $U$ but not on the string $x$. 
Kolmogorov Complexity

**Upper Bound 2**

We have the following upper bound on $K_U(x)$:

$$K_U(x) \leq |x| + 2 \log_2 |x| + C$$

for some constant $C$ which depends on the computer $U$ but not on the string $x$.

**Proof:** Let $q$ be the program:

read integer $n$ and print the following $n$ bits:

$$n_1 1n_2 1 \ldots n_{|n|} 0 x_1 x_2 \ldots x_n$$

The length of $n = |x|$ is at most $\lceil \log_2 |x| \rceil \leq \log_2 |x| + 1$, so that the length of the program is at most $C' + 2 \log_2 |x| + 2 + |x|$.  \[\square\]
Conditional Kolmogorov Complexity

The conditional Kolmogorov complexity is defined as the length of the shortest program to print $x$ when $y$ is given:

$$K_U(x \mid y) = \min_{p : U(\bar{y} \ p) = x} |p|,$$

where $\bar{y}$ is a ‘self-delimiting’ representation of $y$. 

Upper Bound 3

We have the following upper bound on $K_U(x \mid |x|)$:

$$K_U(x \mid |x|) \leq |x| + C$$

for some constant $C$ independent of $x$. 

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Conditional Kolmogorov Complexity

The **conditional Kolmogorov complexity** is defined as the length of the shortest program to print \( x \) when \( y \) is given:

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K_U(x \mid y) = \min_{p : U(\tilde{y} p) = x} |p|
\]

where \( \tilde{y} \) is a ‘self-delimiting’ representation of \( y \).

Upper Bound 3

We have the following upper bound on \( K_U(x \mid |x|) \):

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for some constant \( C \) independent of \( x \).
Examples

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1. \( K_U(0101010101\ldots.01 | n) = C. \)

   *Program*: print \( n/2 \) times 01.
Examples

Let $n = |x|$.

1. $K_U(0101010101\ldots| n) = C$.
   
   *Program*: print $n/2$ times 01.

2. $K_U(\pi_1\pi_2\ldots\pi_n| n) = C$.
   
   *Program*: print the first $n$ bits of $\pi$. 
Examples

Let \( n = |x| \).

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   *Program*: print the first \( n \) bits of \( \pi \).

3. \( K_U(\text{English text} \mid n) \approx 1.3 \times n + C. \)
   
   *Program*: Huffman code.
   
   (Entropy of English is about 1.3 bits per symbol.)
Examples

Let $n = |x|$.

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   Program: print $n/2$ times 01.

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   Program: print the first $n$ bits of $\pi$.

3. $K_U($English text $| n) \approx 1.3 \times n + C.$
   
   Program: Huffman code.
   (Entropy of English is about 1.3 bits per symbol.)

4. $K_U($fractal$) = C.$
   
   Program: print $\#$ of iterations until $z_{n+1} = z_n^2 + c > T$. 
Examples
Martin-Löf Randomness

Examples (contd.):

5 \( K_U(x \mid n) \approx n \), for almost all \( x \in \{0, 1\}^n \).
Examples (contd.):

5. \( K_U(x | n) \approx n \), for almost all \( x \in \{0, 1\}^n \).

**Proof:** Upper bound \( K_U(x | n) \leq n + C \). Lower bound by a counting argument: less than \( 2^{-k} \) of strings compressible by more than \( k \) bits (Lecture 1).
Examples (contd.):

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**Martin-Löf Randomness**

String \( x \) is said to be **Martin-Löf random** iff \( K_U(x \mid n) \geq n \).
Martin-Löf Randomness

Examples (contd.):

5. $K_U(x \mid n) \approx n$, for almost all $x \in \{0, 1\}^n$.

**Proof:** Upper bound $K_U(x \mid n) \leq n + C$. Lower bound by a counting argument: less than $2^{-k}$ of strings compressible by more than $k$ bits (Lecture 1).

String $x$ is said to be **Martin-Löf random** iff $K_u(x \mid n) \geq n$.

Consequence of point 5 above: An i.i.d. sequence of unbiased coin flips is with high probability Martin-Löf random.
Universal Prediction

Since the set of valid (halting) programs is required to be **prefix-free** we can consider the probability distribution $p^n_U$:

$$p^n_U(x) = \frac{2^{-K_U(x|n)}}{C}, \quad \text{where} \quad C = \sum_{x \in \mathcal{X}^n} 2^{-K_U(x|n)}.$$
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**Universal Probability Distribution**

The distribution $p^n_U$ is universal in the sense that for any other computable distribution $q$, there is a constant $C > 0$ such that 

$$p^n_U(x) \geq C q(x) \quad \text{for all } x \in \mathcal{X}^n.$$
Universal Prediction

Since the set of valid (halting) programs is required to be **prefix-free** we can consider the probability distribution $p_U^n$:

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$$p_U^n(x) \geq C q(x) \text{ for all } x \in \mathcal{X}^n.$$

**Proof idea:** The universal computer $U$ can imitate the Shannon-Fano prefix code with codelengths $\left\lceil \log_2 \frac{1}{q(x)} \right\rceil$. 

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This follows from the relationship between codelengths and probabilities (Kraft!):

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K_U(x) \text{ is small } \Rightarrow p^n_U(x) \text{ is large}
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Universal Prediction

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$$K_U(x) \text{ is small} \implies p^n_U(x) \text{ is large}$$

$$\implies \prod_{i=1}^{n} p^n_U(x_i \mid x_1, \ldots, x_{i-1}) \text{ is large}$$
Universal Prediction

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$$\Rightarrow \prod_{i=1}^{n} p^n_U(x_i \mid x_1, \ldots, x_{i-1}) \text{ is large}$$

$$\Rightarrow p^n_U(x_i \mid x_1, \ldots, x_{i-1}) \text{ is large for most } i \in \{1, \ldots, n\},$$

where $x_i$ denotes the $i$th bit in string $x$. 
Berry Paradox

The smallest integer that cannot be described in ten words?
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Whatever this number is, we have just described (?) it in ten words.
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The smallest uninteresting number?
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The smallest uninteresting number?

Whatever this number is, it is quite interesting!
Non-computability

It is impossible to construct a general procedure (algorithm) to compute $K_U(x)$.

**Non-Computability**

Kolmogorov complexity $K_U : \{0, 1\}^* \rightarrow \mathbb{N}$ is **non-computable**.
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**Proof:** Assume, by way of contradiction, that it would be possible to compute $K_U(x)$. 
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Proof: Assume, by way of contradiction, that it would be possible to compute $K_U(x)$. Then for any $M > 0$, the program

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\text{print a string } x \text{ for which } K_U(x) > M.
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print a string $x$ for which $K_U(x) > M$.
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would print a string with $K_U(x) > M$. A contradiction follows by letting $M$ be larger than the Kolmogorov complexity of this program. Hence, it cannot be possible to compute $K_U(x)$.
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3. MDL Principle
   - Definitions
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   - Prediction & Model Selection
Each string $x \in \{0, 1\}^n$ can be described in two parts:

1. the regular features of $x$, 

$K_U(S|n) + \log_2 |S| + C \geq K_U(x|n)$, where $K_U(S|n)$ is the length of the shortest program to list the members of $S$ (and then halt).
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For set $S \subseteq \{0, 1\}^n$, the length of such a two-part description is

$$K_U(S \mid n) + \log_2 |S| + C \geq K_U(x \mid n),$$

where $K_U(S \mid n)$ is the length of the shortest program to list the members of $S$ (and then halt).
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We can consider $S$ (regular features) a **model**.
Finite Set Models

all strings

Finite Set Models

Structure Function

Minimal Sufficient Statistic

Ideal MDL

Kolmogorov Complexity

Outline

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Three Concepts: Information
Example

For instance, if $x$ is a sequence of biased coin flips, then with high probability the only regular feature is the number of 1s.
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Let \( S_k^n = \{ x \in \{0, 1\}^n : \sum x_i = k \} \). The size of this set is

\[
|S_k^n| = \binom{n}{k} = \frac{n!}{k!(n-k)!} \approx 2^{nH\left(\frac{k}{n}\right)},
\]

where \( H\left(\frac{k}{n}\right) \) is the binary entropy with parameter \( \frac{k}{n} \).
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![Graph of binary entropy](image)
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where $H(\frac{k}{n})$ is the binary entropy with parameter $\frac{k}{n}$.

Thus, the two-part description has length

$$K_U(k) + nH(\frac{k}{n}) + C \geq K_U(x | n).$$
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For instance, if $x$ is a sequence of biased coin flips, then with high probability the only regular feature is the number of 1s.

Let $S^n_k = \{x \in \{0,1\}^n : \sum x_i = k\}$. The size of this set is

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where $H(k/n)$ is the binary entropy with parameter $k/n$.

Thus, the two-part description has length

$$K_U(k) + nH(k/n) + C \geq K_U(x \mid n).$$

By the Asymptotic Equipartition Property (Lecture 3), $nH(k/n)$ is with high probability a lower bound ($\approx$) on $K_U(x \mid n)$. 
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Thus, the two-part description has length

\[
K_U(k) + nH\left(\frac{k}{n}\right) + C \approx K_U(x \mid n).
\]

By the **Asymptotic Equipartition Property** (Lecture 3), \( nH\left(\frac{k}{n}\right) \) is with high probability a lower bound (\( \approx \)) on \( K_U(x \mid n) \).
Kolmogorov Structure Function

The **Kolmogorov structure function** is defined as

\[
h_x(\alpha) = \min_{S : x \in S} \log_2 |S|, \quad K_U(S|n) \leq \alpha
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i.e., the log-size of the smallest set containing \( x \) that can be described in \( \alpha \) bits.
### Kolmogorov Structure Function

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i.e., the log-size of the smallest set containing \(x\) that can be described in \(\alpha\) bits.

The more bits we can use to describe \(S\), the more regularities we can cover, which makes \(|S|\) smaller.
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\[ h_x(\alpha) = \min_{S : x \in S} \log_2 |S|, \quad K_U(S|n) \leq \alpha \]

i.e., the log-size of the smallest set containing \( x \) that can be described in \( \alpha \) bits.

The more bits we can use to describe \( S \), the more regularities we can cover, which makes \( |S| \) smaller.

For all \( \alpha > 0 \), there is a two-part description of length \( \alpha + h_x(\alpha) \).
Consider different values of $\alpha$:

$\alpha \approx 0$:

We can only describe the whole set $\{0, 1\}^n$, and not much else, so that $h_x(0) = \log_2 |\{0, 1\}^n| = n$. 

$\alpha \approx K_U(x|n)$:

We can use the singleton set $S = \{x\}$ since $K_U(\{x\}|n) = K_U(x|n) + C$. Thus, $h_x(K_U(x)) = \log_2 |\{x\}| = 0$.

$0 < \alpha < K_U(x|n)$:

A two-part description can never be better than optimal.
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A two-part description can never be better than optimal:

$$h_x(\alpha) \geq K_U(x \mid n) - \alpha$$
The **slope** of the structure function \( h_x(\alpha) \) is at least as steep as \(-1\) (ignoring constants):
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For $k$ extra bits in $\alpha$, we can reduce the set $S$ in a fraction of $\frac{1}{2^k}$ by sorting $S$ alphabetically, dividing in $2^k$ equal size parts, and encoding the index of the part including $x$. 
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The constants related to instructions like “sort alphabetically and choose second half/third quarter/etc.” cause **bumps** in the structure function.
Structure Function

[Adapted from (Vereshchagin & Vitányi, 2004)]
Sufficient Statistic

In statistics, a **sufficient statistic** is a function of the data which contains all the information relevant to a parameter. Examples:

1. In coin flipping, the number of 1s is sufficient for the bias parameter.
2. In die tossing, the number of times each face is seen is sufficient for the parameters $p_1, \ldots, p_6$.
3. In a Gaussian density, the average $\frac{1}{n} \sum X_i$ is sufficient for the mean.
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Three Concepts: Information
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In all these, the order of the outcomes, for instance, is irrelevant.

When estimating the parameter, it is *sufficient* to know the sufficient statistic.
A finite set \( S \) is a **Kolmogorov sufficient statistic** iff we have

\[
K_U(S \mid n) + \log_2 |S| = K_U(x \mid n) + C,
\]

i.e., the two-part description using \( S \) is optimal.
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A Kolmogorov sufficient statistic tells everything about the *structure* of the data $x$. 

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Structure Function

\[ K_U(x) \]

\[ h_x(\alpha) \]

\[ \alpha \]

\[ K_U(x) \]

[Adapted from (Vereshchagin & Vitányi, 2004)]
Minimal Sufficient Statistic

If $S$ is a Kolmogorov sufficient statistic, i.e., we have

$$K_U(S \mid n) + \log_2 |S| = K_U(x \mid n),$$

then for all $\alpha$ within the range $K_U(S \mid n) < \alpha \leq K_U(x \mid n)$, there is another sufficient statistic with complexity $\alpha$:

$$\alpha = K_U(S \mid n) + k : K_U(S \mid n) + k + \log_2 \frac{|S|}{2^k} = K_U(x \mid n).$$
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**Kolmogorov Minimal Sufficient Statistic**

The least complex Kolmogorov sufficient statistic is called the **Kolmogorov minimal sufficient statistic**. It contains all the information about the *structure* of $x$ but nothing more.
Structure Function

[Adapted from (Vereshchagin & Vitányi, 2004)]
Example: Mona Lisa

Figure 7.6. Mona Lisa.

Source: Cover & Thomas, 1991
Example: Mona Lisa
Ideal MDL

Given data $x$, choosing the Kolmogorov minimal sufficient statistic as the preferred model is called "ideal MDL".
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Extracts all regularity from data, and leaves out noise.
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Extracts all regularity from data, and leaves out noise.

\[
\text{Complexity} = \text{Information} + \text{Noise} = \text{Regularity} + \text{Randomness} = \text{Algorithm} + \text{Compressed file}
\]
The finite set models can be replaced by (computable) probability distributions — distribution $p$ is a *sufficient statistic* iff

$$K_U(p \mid n) + \log_2 \frac{1}{p(x)} = K_U(x \mid n),$$

i.e., two-part code optimal.
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i.e., two-part code optimal.

All things essentially unchanged:
- Finite set $S$ can be replaced by a uniform distribution over $S$,
- Distribution $p$ can be replaced by the typical set under $p$. 
1. Kolmogorov Complexity
   - Definition
   - Basic Properties

2. Structure Function
   - Finite Set Models
   - Structure Function
   - Minimal Sufficient Statistic
   - Ideal MDL

3. MDL Principle
   - Definitions
   - Universal Models
   - Prediction & Model Selection
Ideal vs. Practical MDL

There are two problematic issues with ideal MDL:

1. Uncomputability of Kolmogorov complexity.
2. Hidden constants in the definitions and theorems.

⇒ Says nothing about individual strings.

Practical MDL aims to solve these issues by:

1. Replace computer programs by probabilistic models. ⇒ Computable.
2. Replace universal computer $U$ by a universal model. ⇒ No hidden constants.
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**Practical MDL** aims to solve these issues by:

1. Replace computer programs by probabilistic models.
   ⇒ Computable.
2. Replace universal computer $U$ by a **universal model**.
   ⇒ No hidden constants.
Definitions

We call a probability distribution $p : \mathcal{D} \to [0, 1]$ a **model**.

A **model class** $\mathcal{M} = \{ p_\theta : \theta \in \Theta \}$ is a set of probability distributions (models).
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The model within \( \mathcal{M} \) that achieves the shortest codelength for data \( x \) is the **maximum likelihood (ML) model**:

\[
\min_{\theta \in \Theta} \log_2 \frac{1}{p_\theta(D)} = \log_2 \frac{1}{p_{\hat{\theta}}(D)}.
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$$

Depends on $D$!

For model $q$, the excess codelength or “**regret**” over the ML model in $\mathcal{M}$ is given by

$$
\log_2 \frac{1}{q(D)} - \log_2 \frac{1}{p_\hat{\theta}(D)}.
$$
A model (code) for which the regret grows slower than $n$ is said to be a **universal model** (code) relative to model class $\mathcal{M}$:

$$\lim_{n \to \infty} \frac{1}{n} \left[ \log_2 \frac{1}{q(D)} - \log_2 \frac{1}{p^\theta(D)} \right] = 0.$$
A model (code) for which the regret grows slower than $n$ is said to be a **universal model** (code) relative to model class $\mathcal{M}$:

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The universal computer $U$ is universal relative to all computers, while a universal model is universal relative to a model class $\mathcal{M}$. 
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The universal computer $U$ is universal relative to all computers, while a universal model is universal relative to a model class $\mathcal{M}$.

**Stochastic Complexity**

The **stochastic complexity** of data $D$ relative to model class $\mathcal{M}$ is the codelength achieved by a universal model.
The **two-part code** (Lecture 5) consists of

1. optimally quantized parameter values $\theta^q$, and
2. encoding of the data under model $p_{\theta^q}$. 

For 'smooth' parametric models, and optimal quantization, the codelength becomes ($\approx$)

$$\log_2 \frac{1}{p_{\hat{\theta}}(D)} + k \frac{2}{2} \log_2 n,$$

so that the regret is $k \frac{2}{2} \log_2 n$. Since $\log_2 n$ grows slower than $n$, the two-part code is universal.
Two-Part Universal Model

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The total codelength is

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\log_2 \frac{1}{p_{\theta^q}(D)} + \ell(\theta^q)
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so that the regret is $\frac{k}{2} \log_2 n$. Since $\log_2 n$ grows slower than $n$, the two-part code is universal.
There are universal codes that are strictly better than the two-part code.
Mixture Universal Model

There are universal codes that are strictly better than the two-part code.

For instance, given a code for the parameters, let \( w \) be a distribution over the parameter space \( \Theta \) (quantized if necessary) defined as

\[
w(\theta) = \frac{2^{-\ell(\theta)}}{c}, \quad \text{where } c = \sum_{\theta \in \Theta} 2^{-\ell(\theta)}.
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\]

Let \( p^w \) be a mixture distribution over the data-sets \( D \in \mathcal{D} \), defined as

\[
p^w(D) = \sum_{\theta \in \Theta} p_{\theta}(D) \ w(\theta),
\]

i.e., an “average” distribution, where each \( p \) is weighted by \( w \).
The code length of the mixture model $p^w$ is given by

$$\log_2 \frac{1}{\sum_{\theta \in \Theta} p(D \mid \theta) w(\theta)} \leq \log_2 \frac{1}{\max_{\theta \in \Theta} p(D \mid \theta) w(\theta)}$$

$$= \log_2 \frac{1}{\max_{\theta \in \Theta} p(D \mid \theta)} + \log_2 \frac{c}{2^{-\ell(\theta)}}.$$
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= \log_2 \frac{1}{\max_{\theta \in \Theta} p(D \mid \theta)} + \log_2 \frac{c}{2^{-\ell(\theta)}}.
$$

The right-hand side is equal to

$$
\underbrace{\log_2 \frac{1}{p(\hat{\theta}(D))}}_{\text{two-part code}} + \ell(\theta) - \log_2 \frac{1}{c} \leq 0.
$$

The mixture code is always at least as good as the two-part code.
Mixture Universal Model

The codelength of the mixture model $p^w$ is given by

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\log_2 \frac{1}{\sum_{\theta \in \Theta} p(D | \theta) w(\theta)} \leq \log_2 \frac{1}{\max_{\theta \in \Theta} p(D | \theta) w(\theta)} = \log_2 \frac{1}{\max_{\theta \in \Theta} p(D | \theta)} + \log_2 c 2^{-\ell(\theta)}. 
$$

The right-hand side is equal to

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\log_2 \frac{1}{p_\hat{\theta}(D)} + \ell(\theta) - \log_2 \frac{1}{c},
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The mixture code is always at least as good as the two-part code.
Normalized Maximum Likelihood

Consider the maximum likelihood model

\[ p_\hat{\theta}(D) = \max_{\theta \in \Theta} p_{\theta}(D) . \]

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Normalized Maximum Likelihood

Consider the maximum likelihood model

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It is the best probability assignment achievable under model \( \mathcal{M} \).

Unfortunately, it is not possible to use the ML model for coding because is not a probability distribution, i.e.,

\[ C = \sum_{D \in \mathcal{D}} p_{\hat{\theta}}(D) > 1 , \]

unless \( \hat{\theta} \) is constant wrt. \( D \).
The normalized maximum likelihood (NML) model is obtained by normalizing the ML model:

\[ p_{\text{nml}}(D) = \frac{p_{\theta}(D)}{C}, \quad \text{where} \quad C = \sum_{D \in \mathcal{D}} p_{\theta}(D). \]
The normalized maximum likelihood (NML) model is obtained by normalizing the ML model:

\[ p_{\text{nml}}(D) = \frac{p_\hat{\theta}(D)}{C} , \quad \text{where} \quad C = \sum_{D \in \mathcal{D}} p_\hat{\theta}(D) . \]

The regret of NML is given by

\[ \log_2 \frac{1}{p_{\text{nml}}(D)} - \log_2 \frac{1}{p_\hat{\theta}(D)} = \log_2 \frac{C}{p_\hat{\theta}(D)} - \log_2 \frac{1}{p_\hat{\theta}(D)} = \log_2 C , \]

which is constant wrt. \( D \).
Normalized Maximum Likelihood

Let \( q \) be any distribution other than \( p_{\text{nml}} \). Then

- there must a data-set \( D' \in D \) for which we have

\[
q(D') < p_{\text{nml}}(D')
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Let $q$ be any distribution other than $p_{nml}$. Then

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\[\iff \log_2 \frac{1}{q(D')} - \log_2 \frac{1}{p_{\hat{\theta}}(D')} > \log_2 \frac{1}{p_{nml}(D')} - \log_2 \frac{1}{p_{\hat{\theta}}(D')},\]

regret of $q$

regret of $p_{nml}$
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For $D'$, the regret of $q$ is greater than $\log_2 C$, the regret of $p_{nml}$. 
Normalized Maximum Likelihood

Let $q$ be any distribution other than $p_{\text{nml}}$. Then

- there must a data-set $D' \in D$ for which we have

$$q(D') < p_{\text{nml}}(D')$$

$$\Leftrightarrow \log_2 \frac{1}{q(D')} - \log_2 \frac{1}{p_{\hat{\theta}}(D')} > \log_2 \frac{1}{p_{\text{nml}}(D')} - \log_2 \frac{1}{p_{\hat{\theta}}(D')}$$

For $D'$, the regret of $q$ is greater than $\log_2 C$, the regret of $p_{\text{nml}}$.

Thus, the worst-case regret of $q$ is greater than the (worst-case) regret of NML. $\Rightarrow$ NML has the least possible worst-case regret.
Universal Models

For ‘smooth’ parametric models, the regret of NML, \( \log_2 C \), grows slower than \( n \), so **NML is also a universal model.**
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We have seen three kinds of universal models:

1. two-part,
2. mixture,
3. NML.
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We can use them for (at least) three purposes:

1. compression,
2. prediction,
3. model selection.
Prediction is done like in Kolmogorov complexity: universal probability distribution/universal model achieves

- **good compression**: $\ell(D)$ is small,
- **good predictions**: $p(D_i \mid D_1, \ldots, D_{i-1})$ is large for most $i \in \{1, \ldots, n\}$.
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For instance, the mixture code gives a natural predictor which is equivalent to **Bayesian prediction**.
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For instance, the mixture code gives a natural predictor which is equivalent to **Bayesian prediction**.

The NML model gives predictions that are good relative to the best model in the model class, **no matter what happens**.
MDL Model Selection

Recall (from Lecture 5) the multi-part codes used when multiple model classes, $\mathcal{M}_1, \mathcal{M}_2, \ldots$ are available:

1. Encoding of the model class index: $C_0(i), i \in \mathbb{N}$.
2. Encoding of the parameter (vector): $C_i(\theta), \theta \in \Theta_i$.
3. Encoding of the data: $C_{\theta}(D), D \in D$.

If we are interested in choosing a model class (and not the parameters), we can improve parts 2 & 3 by combining them into a better universal code than two-part:

1. Encoding of the model class index: $C_0(i), i \in \mathbb{N}$.
2. Encoding of the data: $C_{M_i}(D), D \in D$, where $C_{M_i}$ is a universal code (e.g., mixture, NML) based on model class $\mathcal{M}_i$. 
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1. Encoding of the model class index: $C_0(i), i \in \mathbb{N}$.
2. Encoding of the data: $C_{\mathcal{M}_i}(D), D \in \mathcal{D}$, where $C_{\mathcal{M}_i}$ is a universal code (e.g., mixture, NML) based on model class $\mathcal{M}_i$. 
The idea is the same as in the Kolmogorov minimal sufficient statistic (ideal MDL): \textbf{Extract all the structure from the data.}
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**MDL Explanation of MDL**

The success in extracting the structure from data can be measured by the codelength.
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**MDL Explanation of MDL**

The success in extracting the structure from data can be measured by the codelength.

In practical MDL, we only find the structure that is ‘visible’ to the used model class(es). For instance, the Bernoulli (coin flipping) model only sees the number of 1s.
The MDL model selection criterion

\[
\text{minimize } \ell(\theta) + \ell_\theta(D)
\]

can be interpreted (via \( p = 2^{-\ell} \)) as

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\text{maximize } p(\theta) \times p_\theta(D).
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MDL & Bayes

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In Bayesian probability, this is equivalent to **maximization of posterior probability**:

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p(\theta \mid D) = \frac{p(\theta) p(D \mid \theta)}{p(D)} ,
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where the term \( p(D) \) (the marginal probability of \( D \)) is constant wrt. \( \theta \) and doesn’t affect model selection.
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\[\Rightarrow \text{ Three Concepts: Probability} \]
Example: Denoising


given

\[ \text{Complexity} = \text{Information} + \text{Noise} \]
\[ = \text{Regularity} + \text{Randomness} \]
\[ = \text{Algorithm} + \text{Compressed file} \]
Example: Denoising

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Denoising means the process of removing noise from a signal.
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The MDL principle gives a natural method for denoising since the very idea of MDL is to separate the total complexity of a signal into information and noise.
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**Denoising** means the process of removing noise from a signal.

The MDL principle gives a natural method for denoising since the very idea of MDL is to separate the total complexity of a signal into information and noise.

First encode a smooth signal (information), and then the difference to the observed signal (noise).
Example: Denoising

| Noisy | PSNR=19.8 | MDL (A-B) | PSNR=32.9 |

Teemu Roos
Three Concepts: Information
Example: Denoising
Example: Denoising
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![Image of a denoised image with a central dark area surrounded by orange and red pixels.](image-url)
The End.