Information-Theoretic Modeling

Lecture 7: Source Coding: Practice (continued)

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Fall 2009

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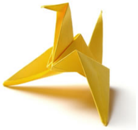
Concentric Circular Tower
(David Huffman)

[Photo: Tony Grant. Courtesy of the Huffman family.]

"Design with the help of binary code (0 and 1) the most efficient method to represent characters, figures and symbols."

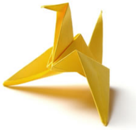
(Assignment at Prof. R.M. Fano's 1952 MIT Information Theory course.)

1. Symbol Codes
    - Entropy lower bound
    - Shannon-Fano Coding
    - Huffman



1. Symbol Codes
   - Entropy lower bound
   - Shannon-Fano Coding
   - Huffman

2. Beyond Symbols Codes
   - Problems with Symbol Codes
   - Two-Part Codes
   - Block Codes
   - Arithmetic Coding



Entropy Lower Bound

So what have we learned?

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1. $E[\ell(X)] - H(X) = D(p \parallel q) + \log_2 \frac{1}{c}$, where $q(x) = \dfrac{2^{-\ell(x)}}{c}$.

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Note also that for a sequence $X_1, \ldots, X_n$ the expected codelength becomes

$$E[\ell(X_1, \ldots, X_n)] = E\left[\sum_{i=1}^{n} \ell(X_i)\right]$$

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! By Shannon's Noiseless Channel Coding Theorem, this is optimal among all codes, **not only symbol codes.**

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! By Shannon's Noiseless Channel Coding Theorem, this is optimal among all codes, **not only symbol codes.** Fine print: only if X_i i.i.d.!

Codelengths and Probabilities

The only problem with the $\ell(x) = \log_2 \frac{1}{p(x)}$ codeword choice is the requirement that codeword lengths must be **integers** (try to think about a codeword with length 0.123, for instance), while the so obtained ℓ is not in general an integer.

Codelengths and Probabilities

The only problem with the $\ell(x) = \log_2 \frac{1}{p(x)}$ codeword choice is the requirement that codeword lengths must be **integers** (try to think about a codeword with length 0.123, for instance), while the so obtained ℓ is not in general an integer.

The simplest solution is to round upwards:

Shannon's Code

Given a pmf, the **Shannon code** has the codeword lengths

$$\ell(x) = \left\lceil \log_2 \frac{1}{p(x)} \right\rceil \quad \text{for all } x \in \mathcal{X}.$$

Alice in Wonderland

Shannon's code: Example

	X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$
	a	0.0644	3.9	4
	b	0.0108	6.5	7
	c	0.0178	5.8	6
	d	0.0359	4.7	5
	e	0.0991	3.3	4
	f	0.0147	6.0	7
	g	0.0184	5.7	6
	h	0.0535	4.2	5
	i	0.0551	4.1	5
	j	0.0011	9.8	10
	k	0.0083	6.8	7
	l	0.0343	4.8	5
		⋮		
	y	0.0165	5.9	6
	z	0.0005	10.7	11
		0.2111	2.2	3

$$H(X) = 4.03$$

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Shannon (1948):

1. Sort by probability.

Shannon's code: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$
	0.2111	2.2	3
e	0.0991	3.3	4
t	0.0781	3.6	4
a	0.0644	3.9	4
o	0.0598	4.0	5
i	0.0551	4.1	5
h	0.0535	4.2	5
n	0.0516	4.2	5
s	0.0475	4.3	5
r	0.0401	4.6	5
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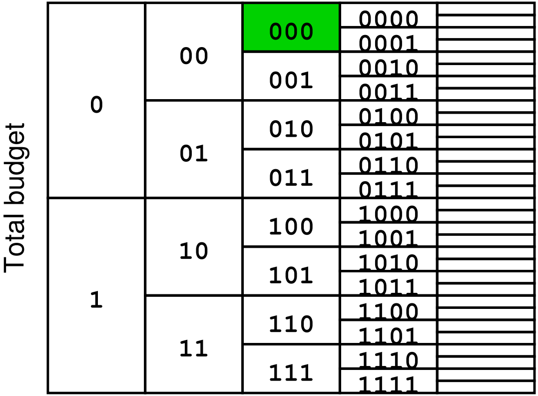
d	0.0359	4.7	5
l	0.0343	4.8	5
	⋮		
x	0.0011	9.8	10
j	0.0011	9.8	10
z	0.0005	10.7	11

$H(X) = 4.03$

Shannon (1948):

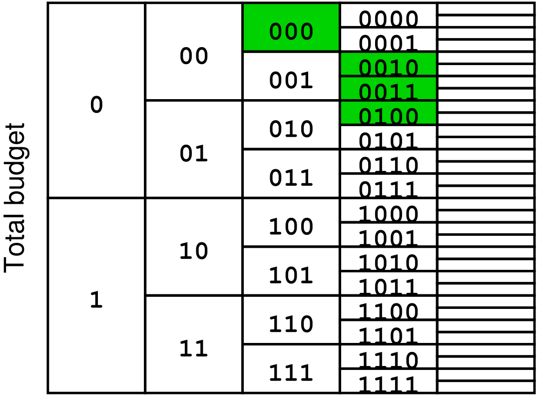
1. Sort by probability.
2. Choose codewords in order, avoiding prefixes. ("Kraft table"!)

Shannon's code: Example



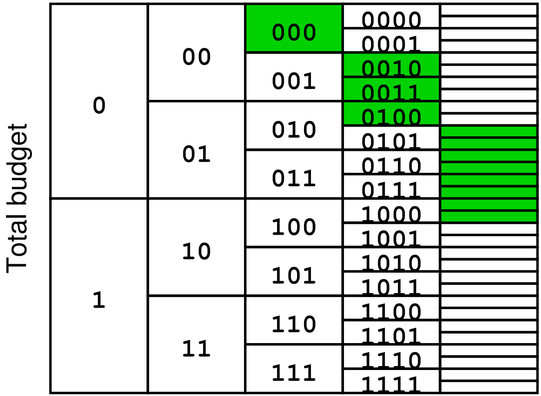
Codeword lengths $({\color{red}3}, 4, 4, 4, 5, 5, 5, 5, \ldots, 10, 10, 11)$

Shannon's code: Example



Codeword lengths $(3, 4, 4, 4, 5, 5, 5, 5, \ldots, 10, 10, 11)$

Shannon's code: Example



Codeword lengths $(3, 4, 4, 4, 5, 5, 5, 5, \ldots, 10, 10, 11)$

Shannon's code: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	3	000
e	0.0991	3.3	4	0010
t	0.0781	3.6	4	0011
a	0.0644	3.9	4	0100
o	0.0598	4.0	5	01010
i	0.0551	4.1	5	01011
h	0.0535	4.2	5	01100
n	0.0516	4.2	5	01101
s	0.0475	4.3	5	01110
r	0.0401	4.6	5	01111
d	0.0359	4.7	5	10000
l	0.0343	4.8	5	10001
	⋮			
x	0.0011	9.8	10	1010111101
j	0.0011	9.8	10	1010111110
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a	0.0644	3.9	4	0100
o	0.0598	4.0	5	01010
i	0.0551	4.1	5	01011
h	0.0535	4.2	5	01100
n	0.0516	4.2	5	01101
s	0.0475	4.3	5	01110
r	0.0401	4.6	5	01111
d	0.0359	4.7	5	10000
l	0.0343	4.8	5	10001
	⋮			
x	0.0011	9.8	10	1010111101
j	0.0011	9.8	10	1010111110
z	0.0005	10.7	11	10101111110

$$H(X) = 4.03$$

$$E[\ell(X)] = 4.60$$

$$E[\ell(X)] - H(X) = 0.57$$

Shannon's code

The expected codeword length of Shannon's code is

$$E\left[\ell(X)\right] = E\left[\left\lceil \log_2 \frac{1}{p(X)} \right\rceil\right] < E\left[\log_2 \frac{1}{p(X)} + 1\right] = H(X) + 1 \ .$$

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In the Alice example we had

$$E[\ell(X)] - H(X) = 4.60 - 4.03 = 0.57 < 1 \ .$$

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$
a	0.0644	3.9
b	0.0108	6.5
c	0.0178	5.8
d	0.0359	4.7
e	0.0991	3.3
f	0.0147	6.0
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y	0.0165	5.9
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(Shannon-)Fano code:

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(Shannon-)Fano code:

1. Sort by probability.

Fano: Example

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	0.2111	2.2
e	0.0991	3.3
t	0.0781	3.6
a	0.0644	3.9
o	0.0598	4.0
i	0.0551	4.1
h	0.0535	4.2
n	0.0516	4.2
s	0.0475	4.3
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X	$p(X)$	$\log_2 \frac{1}{p(X)}$
	0.2111	2.2
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t	0.0781	3.6
a	0.0644	3.9
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(Shannon-)Fano code:

1. Sort by probability.
2. Divide in two equally probable parts (as equal as possible)

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$
	0.2111	2.2
e	0.0991	3.3
t	0.0781	3.6
a	0.0644	3.9
o	0.0598	4.0
i	0.0551	4.1
h	0.0535	4.2
n	0.0516	4.2
s	0.0475	4.3
r	0.0401	4.6
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3. Add '0' to the codewords in the first part, '1' to the others.

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X	$p(X)$	$\log_2 \frac{1}{p(X)}$
	0.2111	2.2
e	0.0991	3.3
t	0.0781	3.6
a	0.0644	3.9
o	0.0598	4.0
i	0.0551	4.1
h	0.0535	4.2
n	0.0516	4.2
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(Shannon-)Fano code:

1. Sort by probability.
2. Divide in two equally probable parts (as equal as possible)
3. Add '0' to the codewords in the first part, '1' to the others.
4. Repeat recursively for both parts.

Fano: Example

	X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
		0.2111	2.2	2	0
	e	0.0991	3.3	4	0
	t	0.0781	3.6	4	0
	a	0.0644	3.9	4	0
	o	0.0598	4.0	4	0
	i	0.0551	4.1	4	1
	h	0.0535	4.2	4	1
	n	0.0516	4.2	4	1
	s	0.0475	4.3	5	1
	r	0.0401	4.6	5	1
	d	0.0359	4.7	5	1
	l	0.0343	4.8	5	1
		⋮			
	x	0.0011	9.8	10	1
	j	0.0011	9.8	10	1
	z	0.0005	10.7	10	1

Fano: Example

	X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
		0.2111	2.2	2	00
	e	0.0991	3.3	4	01
	t	0.0781	3.6	4	01
	a	0.0644	3.9	4	01
	o	0.0598	4.0	4	01
	i	0.0551	4.1	4	10
	h	0.0535	4.2	4	10
	n	0.0516	4.2	4	10
	s	0.0475	4.3	5	10
	r	0.0401	4.6	5	10
	d	0.0359	4.7	5	11
	l	0.0343	4.8	5	11
		⋮			
	x	0.0011	9.8	10	11
	j	0.0011	9.8	10	11
	z	0.0005	10.7	10	11

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X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	00
e	0.0991	3.3	4	010
t	0.0781	3.6	4	010
a	0.0644	3.9	4	011
o	0.0598	4.0	4	011
i	0.0551	4.1	4	100
h	0.0535	4.2	4	100
n	0.0516	4.2	4	101
s	0.0475	4.3	5	101
r	0.0401	4.6	5	101
d	0.0359	4.7	5	110
l	0.0343	4.8	5	110
	⋮			
x	0.0011	9.8	10	111
j	0.0011	9.8	10	111
z	0.0005	10.7	10	111

Fano: Example

X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	00
e	0.0991	3.3	4	0100
t	0.0781	3.6	4	0101
a	0.0644	3.9	4	0110
o	0.0598	4.0	4	0111
i	0.0551	4.1	4	1000
h	0.0535	4.2	4	1001
n	0.0516	4.2	4	1010
s	0.0475	4.3	5	1011
r	0.0401	4.6	5	1011
d	0.0359	4.7	5	1100
l	0.0343	4.8	5	1100
	⋮			
x	0.0011	9.8	10	1111
j	0.0011	9.8	10	1111
z	0.0005	10.7	10	1111

Fano: Example

	X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
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	e	0.0991	3.3	4	0100
	t	0.0781	3.6	4	0101
	a	0.0644	3.9	4	0110
	o	0.0598	4.0	4	0111
	i	0.0551	4.1	4	1000
	h	0.0535	4.2	4	1001
	n	0.0516	4.2	4	1010
	s	0.0475	4.3	5	10110
	r	0.0401	4.6	5	10111
	d	0.0359	4.7	5	11000
	l	0.0343	4.8	5	11001
		⋮			
	x	0.0011	9.8	10	11111
	j	0.0011	9.8	10	11111
	z	0.0005	10.7	10	11111

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a	0.0644	3.9	4	0110
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i	0.0551	4.1	4	1000
h	0.0535	4.2	4	1001
n	0.0516	4.2	4	1010
s	0.0475	4.3	5	10110
r	0.0401	4.6	5	10111
d	0.0359	4.7	5	11000
l	0.0343	4.8	5	11001
	$\vdots$			
x	0.0011	9.8	10	111111
j	0.0011	9.8	10	111111
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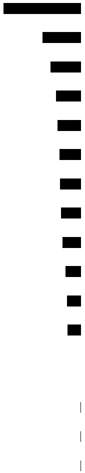
Fano: Example



X	$p(X)$	$\log_2 \frac{1}{p(X)}$	$\ell(X)$	$C(X)$
	0.2111	2.2	2	00
e	0.0991	3.3	4	0100
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$$H(X) = 4.03$$

$$E[\ell(X)] = 4.07$$

$$E[\ell(X)] - H(X) = 0.04$$

Shannon-Fano Code

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$$E\left[\ell(X)\right] \leq H(X) + 1 \ .$$

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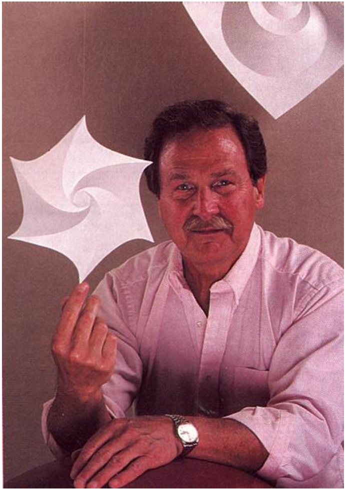
`"Design with the help of binary code (0 and 1) the most efficient method to represent characters, figures and symbols."`

David Huffman (1925–1999)



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See the demo at
`www.cs.auckland.ac.nz/software/AlgAnim/huffman.html`

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Note that since Shannon-Fano gives $E[\ell(X)] \leq H(X) + 1$, and Huffman is optimal, Huffman must satisfy the same bound.

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Usually the overhead is minor compared to the total file size.

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Allows modeling of dependence.

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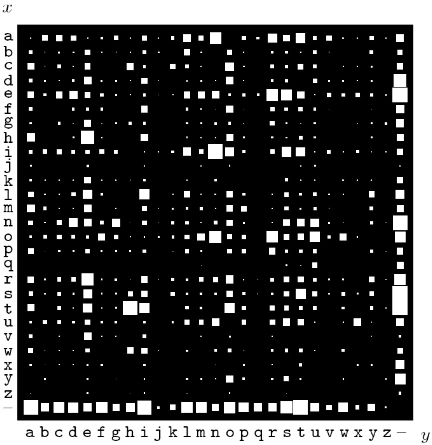
x
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s
t
u
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a b c d e f g h i j k l m n o p q r s t u v w x y z –
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Complexity Tradeoff

Find suitable balance between complexity of the model (increases with N) and codelength of data given model (decreases with N).
$\Rightarrow$ **MDL/MML Principle**

Adaptive Codes

Alternative Solution to Problems 2 & 3:

Adaptive Codes

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Arithmetic coding avoids "all problems": adaptive, spreads the one additional bit over the whole sequence, and can be decoded instantaneously.

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- For a *narrow* interval we need a *long* binary string.

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We continue dividing recursively until the whole source string is encoded.

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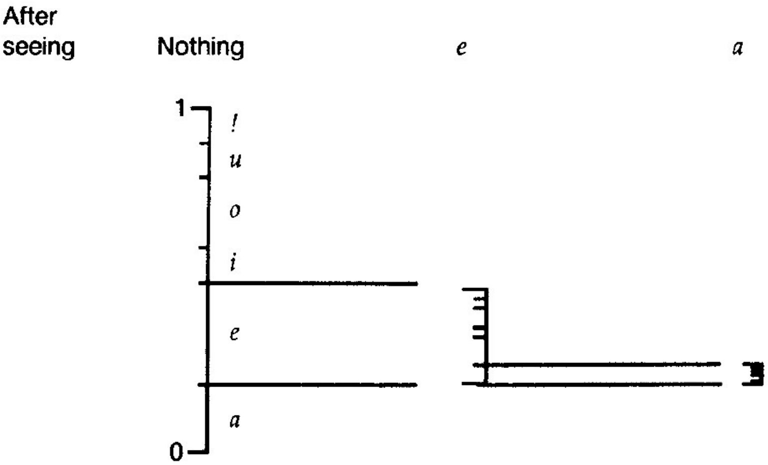

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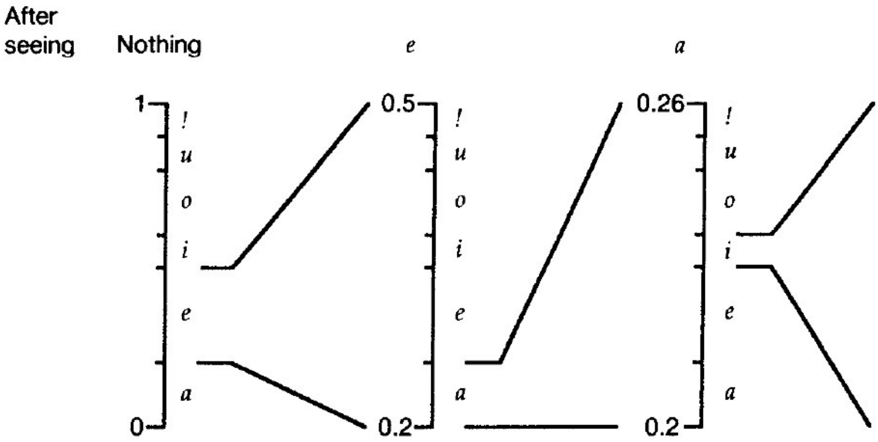

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- Read the material on arithmetic coding (Witten, Neal & Cleary)