Information-Theoretic Modeling

Lecture 9: The MDL Principle

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Lecture 9: MDL Principle

Jorma Rissanen (left) receiving the IEEE Information Theory Society Best Paper Award from Claude Shannon in 1986.

IEEE Golden Jubilee Award for Technological Innovation (for the invention of arithmetic coding) 1998; IEEE Richard W. Hamming Medal (for fundamental contribution to information theory, statistical inference, control theory, and the theory of complexity) 1993; Kolmogorov Medal 2006; IBM Outstanding Innovation Award (for work in statistical inference, information theory, and the theory of complexity) 1988; IEEE Claude E. Shannon Award 2009; ...
1 Occam’s Razor

- House
- Visual Recognition
- Astronomy
- Razor
 Outline
Occam’s Razor
MDL Principle

1 Occam’s Razor
- House
- Visual Recognition
- Astronomy
- Razor

2 MDL Principle
- Rules & Exceptions
- Probabilistic Models
- Old-Style MDL
- Modern MDL
House

Outline
- Occam's Razor
- MDL Principle

Teemu Roos
Information-Theoretic Modeling
Brandon has

1. cough,
2. severe abdominal pain,
3. nausea,
4. low blood pressure,
5. fever.
Brandon has

1. cough,
2. severe abdominal pain,
3. nausea,
4. low blood pressure,
5. fever.

No single disease causes all of these.
Brandon has

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2. severe abdominal pain,
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No single disease causes all of these.

Each symptom can be caused by *some* (possibly different) disease...
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Dr. House explains the symptoms with two simple causes:
Brandon has

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No single disease causes all of these.

Each symptom can be caused by some (possibly different) disease...

Dr. House explains the symptoms with two simple causes:

1. common cold, causing the cough and fever,
Brandon has

1. cough,
2. severe abdominal pain,
3. nausea,
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5. fever.

No single disease causes all of these.

Each symptom can be caused by some (possibly different) disease...

Dr. House explains the symptoms with two simple causes:

1. common cold, causing the cough and fever,
2. pharmacy error: cough medicine replaced by gout medicine.
Visual Recognition
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Visual Recognition
Schema huius præmissæ divisionis Sphærarum.
Astronomy
William of Ockham (c. 1288–1348)
Occam’s Razor

Entities should not be multiplied beyond necessity.
Occam’s Razor

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Isaac Newton: “We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.”
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Diagnostic parsimony: Find the fewest possible causes that explain the symptoms.
Occam’s Razor

Entities should not be multiplied beyond necessity.

Isaac Newton: “We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.”

**Diagnostic parsimony:** Find the fewest possible causes that explain the symptoms.

**Hickam’s dictum:** “Patients can have as many diseases as they damn well please.”
Visual Recognition
Visual Recognition
Visual Recognition
Visual Recognition
1 Occam’s Razor
   - House
   - Visual Recognition
   - Astronomy
   - Razor

2 MDL Principle
   - Rules & Exceptions
   - Probabilistic Models
   - Old-Style MDL
   - Modern MDL
Minimum Description Length (MDL) Principle (2-part)

Choose the hypothesis which minimizes the sum of

1. the codelength of the hypothesis, and
2. the codelength of the data with the help of the hypothesis.
Minimum Description Length (MDL) Principle (2-part)

Choose the hypothesis which minimizes the sum of

1. the codelength of the hypothesis, and
2. the codelength of the data with the help of the hypothesis.

How to encode data \textit{with the help of a hypothesis}?
Idea 1: Hypothesis = rule; encode exceptions.
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Black box of size $25 \times 25 = 625$, white dots at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. 
Encoding Data: Rules & Exceptions

Idea 1: Hypothesis = rule; encode exceptions.

Black box of size $25 \times 25 = 625$, white dots at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

For image of size $n = 625$, there are $2^n$ different images, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

different groups of $k$ exceptions.
Encoding Data: Rules & Exceptions

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For image of size $n = 625$, there are $2^n$ different images, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

different groups of $k$ exceptions.

$k = 1: \binom{n}{1} = 625 \ll 2^{625} \approx 1.4 \times 10^{188}$.

Code length $\log_2(n + 1) + \log_2 \binom{n}{k} \approx 19$ vs. $\log_2 2^{625} = 625$. 

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Encoding Data: Rules & Exceptions

Idea 1: Hypothesis = rule; encode exceptions.

Black box of size $25 \times 25 = 625$, white dots at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

For image of size $n = 625$, there are $2^n$ different images, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

different groups of $k$ exceptions.

$k = 2 : \binom{n}{2} = 195\,000 \ll 2^{625} \approx 1.4 \times 10^{188}$.

Codelength $\log_2(n + 1) + \log_2\binom{n}{k} \approx 27$ vs. $\log_2 2^{625} = 625$
Encoding Data: Rules & Exceptions

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different groups of $k$ exceptions.

$k = 3 : \binom{625}{3} = 40\,495\,000 \ll 2^{625} \approx 1.4 \times 10^{188}$.

Codelength $\log_2(n + 1) + \log_2 \binom{n}{k} \approx 35$ vs. $\log_2 2^{625} = 625$
Encoding Data: Rules & Exceptions

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For image of size $n = 625$, there are $2^n$ different images, and

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different groups of $k$ exceptions.

$k = 10 : \binom{n}{10} = 23313540000000000000 \ll 2^{625}$.

Codelength $\log_2(n + 1) + \log_2 \binom{n}{k} \approx 80$ vs. $\log_2 2^{625} = 625$
Encoding Data: Rules & Exceptions

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For image of size $n = 625$, there are $2^n$ different images, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

different groups of $k$ exceptions.

$k = 100 : \binom{n}{100} \approx 9.5 \times 10^{117} \ll 2^{625} \approx 1.4 \times 10^{188}$.

Codelength $\log_2(n + 1) + \log_2 \binom{n}{k} \approx 401$ vs. $\log_2 2^{625} = 625$
Encoding Data: Rules & Exceptions

**Idea 1:** Hypothesis = rule; encode exceptions.

Black box of size $25 \times 25 = 625$, white dots at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

For image of size $n = 625$, there are $2^n$ different images, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

different groups of $k$ exceptions.

$k = 300 : \binom{n}{300} \approx 2.7 \times 10^{186} < 2^{625} \approx 1.4 \times 10^{188}$.

Codelength $\log_2(n + 1) + \log_2 \binom{n}{k} \approx 629$ vs. $\log_2 2^{625} = 625$
**Idea 1:** Hypothesis = rule; encode exceptions.

Black box of size $25 \times 25 = 625$, white dots at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

For image of size $n = 625$, there are $2^n$ different images, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

different groups of $k$ exceptions.

$k = 372 : \binom{n}{372} \approx 5.1 \times 10^{181} \ll 2^{625} \approx 1.4 \times 10^{188}$. 

Codelength $\log_2(n+1) + \log_2 \binom{n}{k} \approx 613$ vs. $\log_2 2^{625} = 625$
Encoding Data: Probabilistic Models

**Idea 2:** Hypothesis = probability distribution.

- **Noisy**  PSNR=19.8
- **MDL (A-B)**  PSNR=32.9

\[
\text{Rissanen & Shannon: } \log_2 \left( \frac{1}{\hat{p}_\theta(D)} \right) + k \frac{2 \log_2 n}{n}
\]
Encoding Data: Probabilistic Models

**Idea 2:** Hypothesis $= \text{probability distribution.}$

\[
\text{Rissanen & Shannon: } \log_2 \frac{1}{p_\hat{\theta}(D)} + \frac{k}{2} \log_2 n.
\]
Figure 1: A simple (1.1), complex (1.2) and a trade-off (3rd degree) polynomial.

From P. Grünwald
Old-Style MDL

With the precision $\frac{1}{\sqrt{n}}$, the codelength for data is almost optimal:

$$\min_{\theta q \in \{\theta^{(1)}, \theta^{(2)}, \ldots\}} \ell_{\theta q}(D) \approx \min_{\theta \in \Theta} \ell_{\theta}(D) = \log_2 \frac{1}{p_{\hat{\theta}}(D)}.$$
Old-Style MDL

With the precision $\frac{1}{\sqrt{n}}$ the codelength for data is almost optimal:

$$
\min_{\theta^q \in \{\theta^{(1)}, \theta^{(2)}, \ldots\}} \ell_{\theta^q}(D) \approx \min_{\theta \in \Theta} \ell_{\theta}(D) = \log_2 \frac{1}{p_{\theta}(D)}.
$$

This gives the total codelength formula:

**“Steam MDL”**

$$
\ell_{\theta^q}(D) + \ell(\theta^q) \approx \log_2 \frac{1}{p_{\theta}(D)} + \frac{k}{2} \log_2 n.
$$
Old-Style MDL

The $\frac{k}{2} \log_2 n$ formula is only a rough approximation, and works well only for very large samples.
Old-Style MDL

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**MDL in the 21st century:**

- More advanced codes: mixtures, normalized maximum likelihood, etc.
Flavours of MDL

1. "Pedestrian"
Asymptotic two-part code-length same as BIC.

2. "Sophisticated"
Bayesian marginal likelihood.

3. "Champions League"
Modern (minimax regret optimal) code

normalized maximum likelihood (NML)

Problem: NML computationally very hard.
MDL Model Selection

Recall (from Lecture 7) the multi-part codes used when multiple model classes, \( M_1, M_2, \ldots \) are available:

1. Encoding of the model class index: \( \ell(M_i), i \in \mathbb{N} \).
2. Encoding of the parameter (vector): \( \ell_1(\theta), \theta \in \Theta_i \).
3. Encoding of the data: \( \log_2 \frac{1}{p(\theta)(D)}, D \in D \).

If we are interested in choosing a model class (and not the parameters), we can improve parts 2 & 3 by combining them into a better universal code than two-part:

1. Encoding of the model class index: \( \ell(M_i), i \in \mathbb{N} \).
2. Encoding of the data: \( \ell_{M_i}(D), D \in D \), where \( \ell_{M_i} \) is a universal code-length (e.g., mixture, NML) based on model class \( M_i \).
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4. Encoding of the model class index: $\ell(\mathcal{M}_i), \ i \in \mathbb{N}$.
MDL Model Selection

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Information-Theoretic Modeling
The success in extracting the structure from data can be measured by the code length.
MDL Model Selection

MDL Explanation of MDL

The success in extracting the structure from data can be measured by the codelength.

In practice, we only find the structure that is “visible” to the used model class(es). For instance, the Bernoulli (coin flipping) model only sees the number of 1s.
The MDL model selection criterion

\[
\text{minimize } \ell(\theta) + \ell_\theta(D)
\]

can be interpreted (via \( p = 2^{-\ell} \)) as

\[
\text{maximize } p(\theta) \times p_\theta(D).
\]
MDL & Bayes

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can be interpreted (via \( p = 2^{-\ell} \)) as

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\text{maximize } p(\theta) \times p_\theta(D).
\]

In Bayesian probability, this is equivalent to \textbf{maximization of posterior probability}:

\[
p(\theta \mid D) = \frac{p(\theta) p(D \mid \theta)}{p(D)},
\]

where the term \( p(D) \) (the marginal probability of \( D \)) is constant wrt. \( \theta \) and doesn’t affect model selection.
MDL & Bayes

The MDL model selection criterion

$$\text{minimize } \ell(\theta) + \ell_\theta(D)$$

can be interpreted (via $p = 2^{-\ell}$) as

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In Bayesian probability, this is equivalent to maximization of posterior probability:

$$p(\theta | D) = \frac{p(\theta) p(D | \theta)}{p(D)},$$

where the term $p(D)$ (the marginal probability of $D$) is constant wrt. $\theta$ and doesn’t affect model selection.

⇒ Three Concepts: Probability
Example: Denoising

\[
\text{Complexity} = \text{Information} + \text{Noise} = \text{Regularity} + \text{Randomness} = \text{Algorithm} + \text{Compressed file}
\]
Example: Denoising

Complexity = Information + Noise
= Regularity + Randomness
= Algorithm + Compressed file

Denoising means the process of removing noise from a signal.
Example: Denoising

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\text{Complexity} = \text{Information} + \text{Noise} = \text{Regularity} + \text{Randomness} = \text{Algorithm} + \text{Compressed file}
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Denoising means the process of removing noise from a signal.

The MDL principle gives a natural method for denoising since the very idea of MDL is to separate the total complexity of a signal into information and noise.
Example: Denoising

\[ \text{Complexity} = \text{Information} + \text{Noise} \]
\[ = \text{Regularity} + \text{Randomness} \]
\[ = \text{Algorithm} + \text{Compressed file} \]

Denoising means the process of removing noise from a signal.

The MDL principle gives a natural method for denoising since the very idea of MDL is to separate the total complexity of a signal into information and noise.

First encode a smooth signal (information), and then the difference to the observed signal (noise).
Example: Denoising

<table>
<thead>
<tr>
<th>Noisy</th>
<th>PSNR=19.8</th>
<th>MDL (A-B)</th>
<th>PSNR=32.9</th>
</tr>
</thead>
</table>

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Example: Denoising
Example: Denoising
Example: Denoising
Example: Denoising
Example: Denoising
Friday’s lecture:

- Real examples of MDL in action.