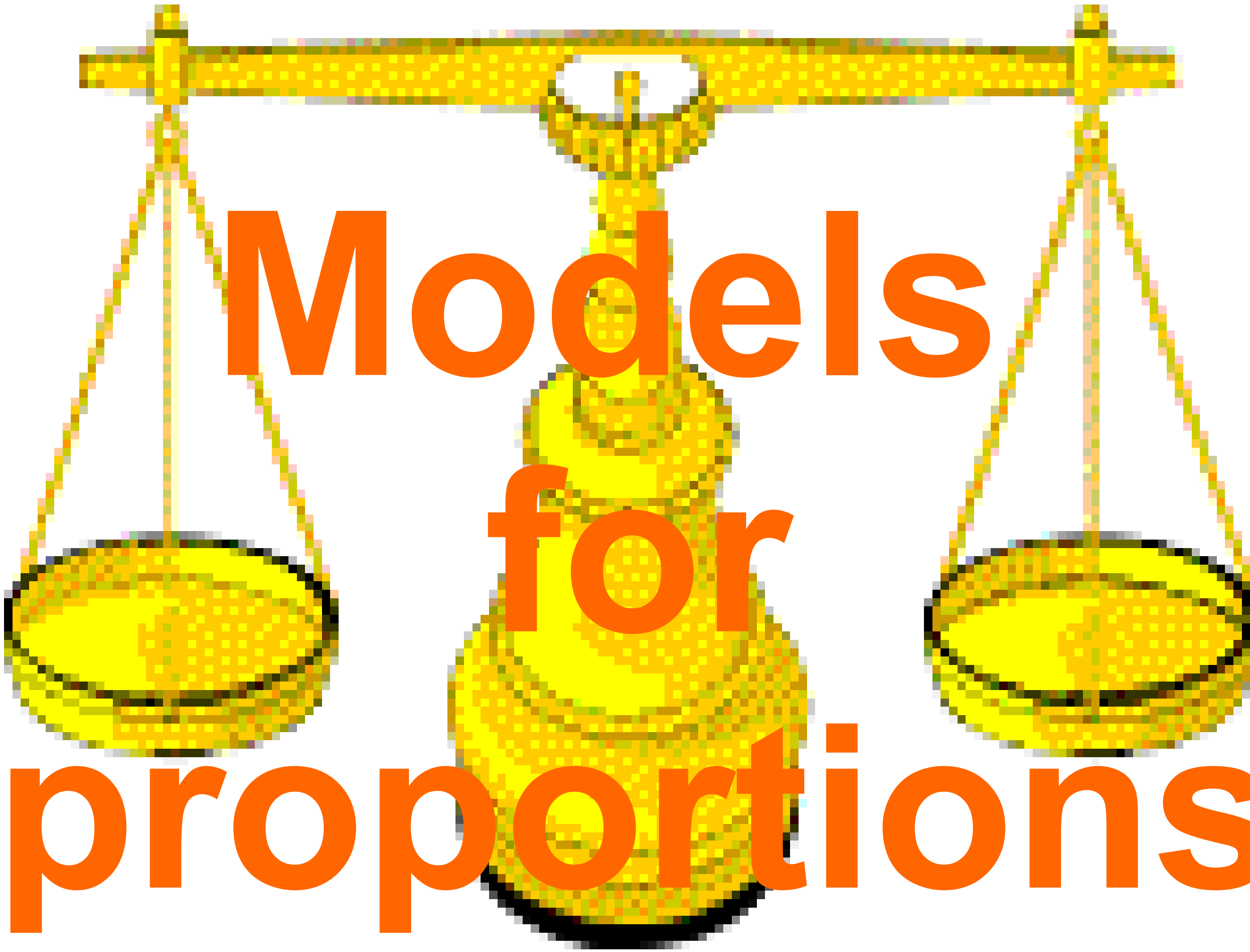
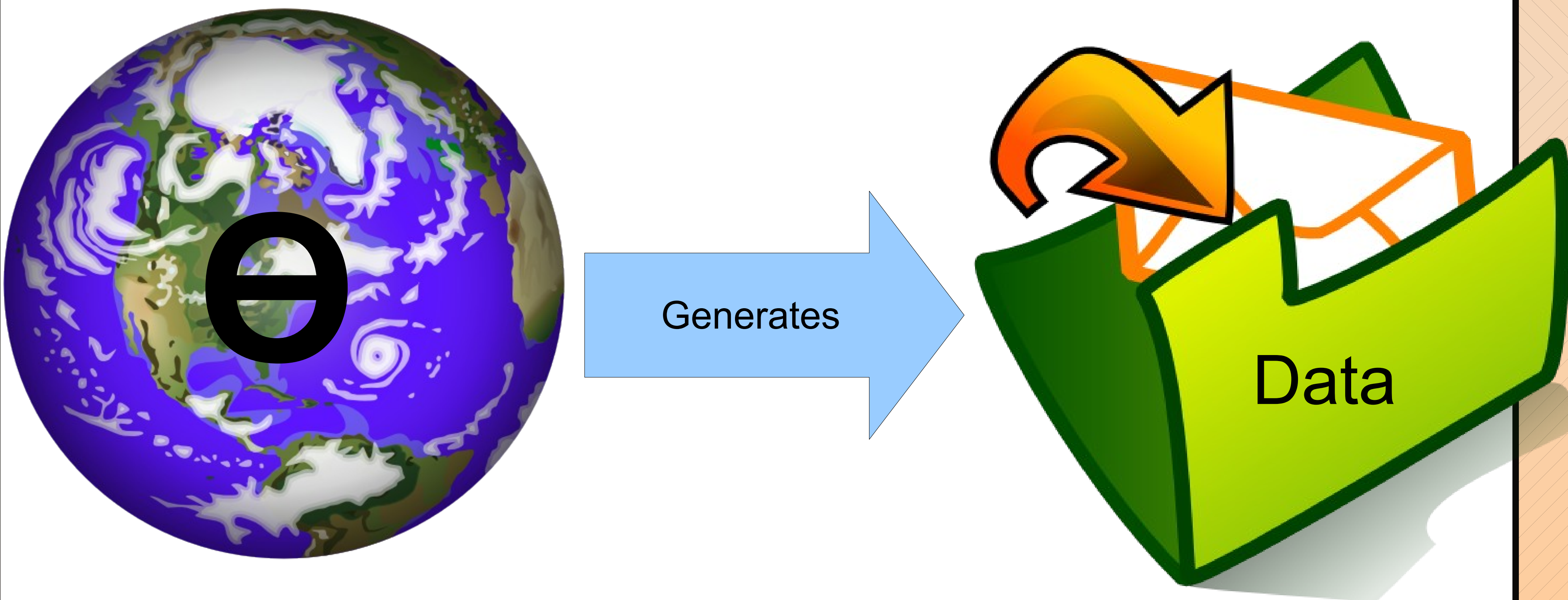


Models for proportions

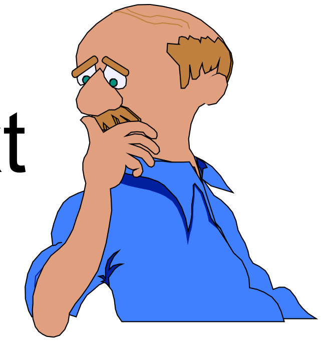
Generative model

- The world is described by a model that governs the probabilities of observing different kinds of data.



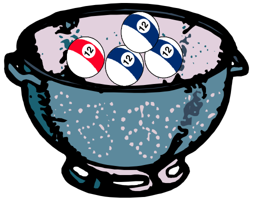
Steps in Bayesian inference

- Specify a set of generative probabilistic models
- Assign a prior probability to each model
- Collect data
- Calculate the likelihood P(data|model) of each model
- Use Bayes' rule to calculate the posterior probabilities P(model | data)
- Draw inferences (e.g., predict the next observation)



Likelihood P(d|Θ)

- Data item d is generated by a mechanism (model), parameters Θ of which determine how probably different values of d are generated, i.e., the distribution of d.
- An example:



  - Mechanism is drawing with replacement from a bucket of black and white balls, and the parameter θ_b is the number of black balls, and the θ_w is the number of white balls in a bucket:
    - $P(b|\theta_b,\theta_w) = \theta_b/(\theta_b+\theta_w)$ and $P(w|\theta_b,\theta_w) = \theta_w/(\theta_b+\theta_w)$.
- In orthodox statistics, likelihood P(D|Θ) is often seen as a function of Θ, a kind of $L_D(\Theta)$. Whatever.

i.i.d.

- If the data generating mechanism depends on ϴ only (and not on what has been generated before), the sequence of data data is called *independent* and *identically distributed*.
- Then $$P(d_{1,} d_{2,} \ldots, d_n | \theta) = \prod_{i=1}^{n} P(d_i | \theta)$$
- And
  - order of d_i does not matter.
  - $P(b, w, b, b, w | \theta) = P(b, b, w, w, w | \theta)$
    $= P(b|\theta) P(b|\theta) P(w|\theta) P(w|\theta) P(w|\theta)$

The Bernoulli model

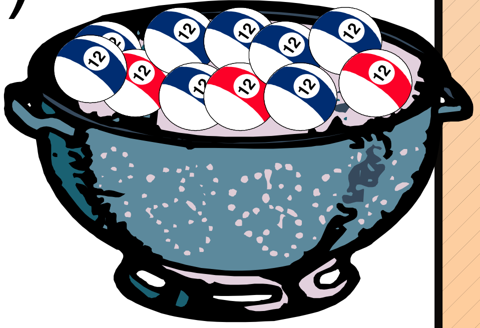
- A model for i.i.d. binary outcomes (heads,tails), (1,0), (black, white), (true, false),....
- One parameter: $\Theta \in [0,1]$. For example: $P(d=true \mid \Theta) = \Theta$, $P(d=false \mid \Theta) = 1-\Theta$.
  - NB! The probabilities of d being true are **defined by** the parameter Θ. Parameters are not probabilities.
  - Black and white ball bucket as a Bernoulli model:
    - Θ is the proportion of black balls in a bucket $P(b \mid \Theta) = \Theta$.
    - $P(D|\Theta) = \Theta^{Nb} (1-\Theta)^{Nw}$, where N_b and N_w are numbers of black and white balls in the data D.
    - NB! $P(D|\Theta)$ depends on data D through N_b and N_w only (=sufficient statistics)

Example

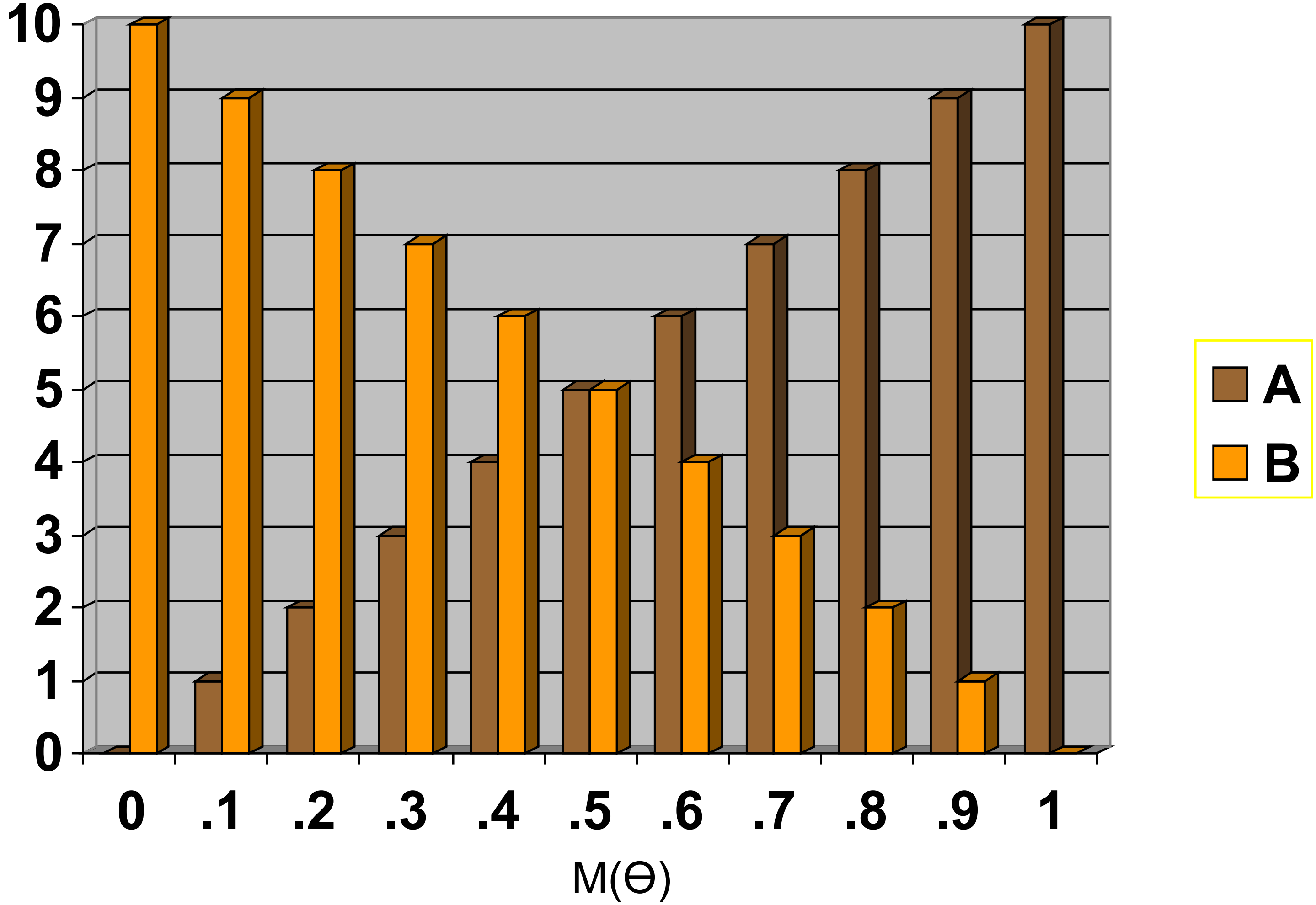
- You are installing WLAN-cards for different machines. You get the WLAN-cards from the same manufacturer, and some of them are faulty.
- We are asking the question: “Is the next WLAN-card we are installing going to work?”
- We are allowed to have background knowledge of these cards (they have been reliable/unreliable in the past, the manufacturing quality has gone up/down etc.)

Assessing models

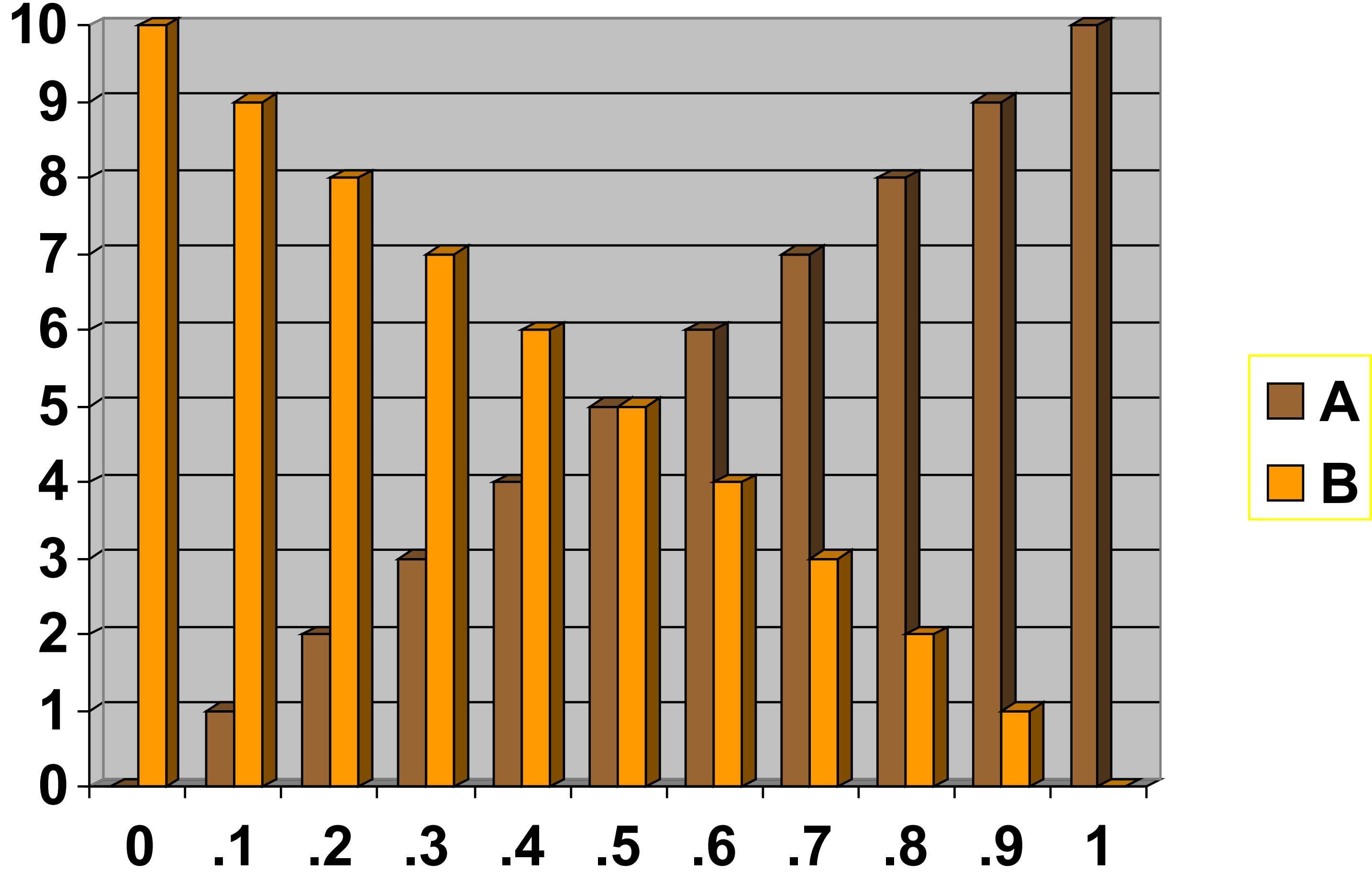
- Let A = “The WLAN-card is not faulty”, and B=~A
- A proportion model can be understood as a bowl with labeled balls (A,B)
- each model $M(\Theta)$ is characterized by the number of A balls, Θ is the proportion (Obs! Assume here that Θ is discrete, i.e., only consider $\Theta \in \{0,0.1,0.2,\ldots,1\}$)



Our 11 models

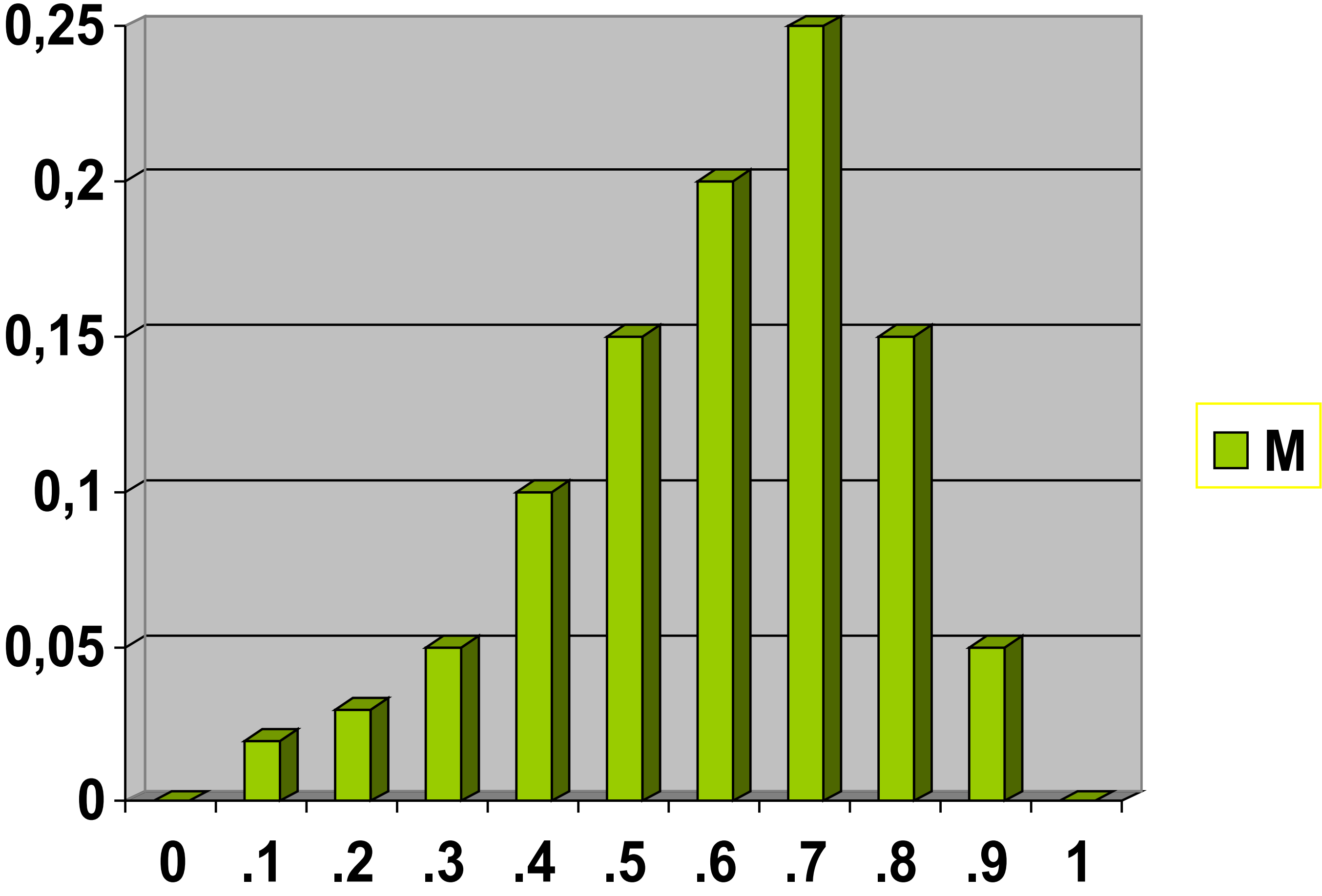


Priors and the models



M(Θ)	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
P(M(Θ))	0	0.02	0.03	0.05	0.1	0.15	0.2	0.25	0.15	0.05	0

The prior distribution P(M(ϴ))



Prediction by model averaging

- A Bayesian predicts by model averaging: the uncertainty about the model is taken into account by weighting the predictions of the different alternative models M_i (=*marginalization* over the unknown)

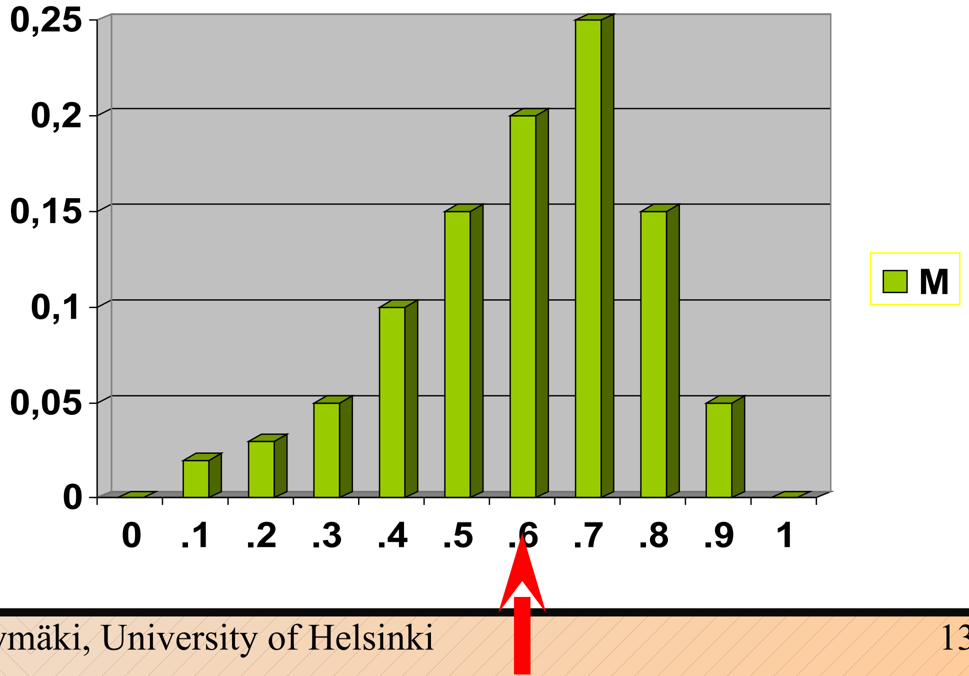
$$P(X) = \sum_i P(X|M_i)P(M_i)$$

So: the predictive probability is...

- What is P(A), the probability that the next WLAN-card is not faulty?

$$P(A) = P(A|M(0.0))P(M(0.0)) + P(A|M(0.1))P(M(0.1)) + \ldots + P(A|M(1.0))P(M(1.0))$$
$$= 0.0 + 0.02 + 0.03 + \ldots + 0.0 = 0.598$$

- ”Mean or average” model: ϴ =0.598
- 60/40 odds a priori



Enter some data ...

- Assume that I have installed three WLAN-cards: first was non-faulty (A), the two latter ones faulty (B), i.e., D={ABB}
- what are the updated (posterior) probabilities for the models M(Θ)?
- Enter Bayes, for example for M(0.6):

0.2

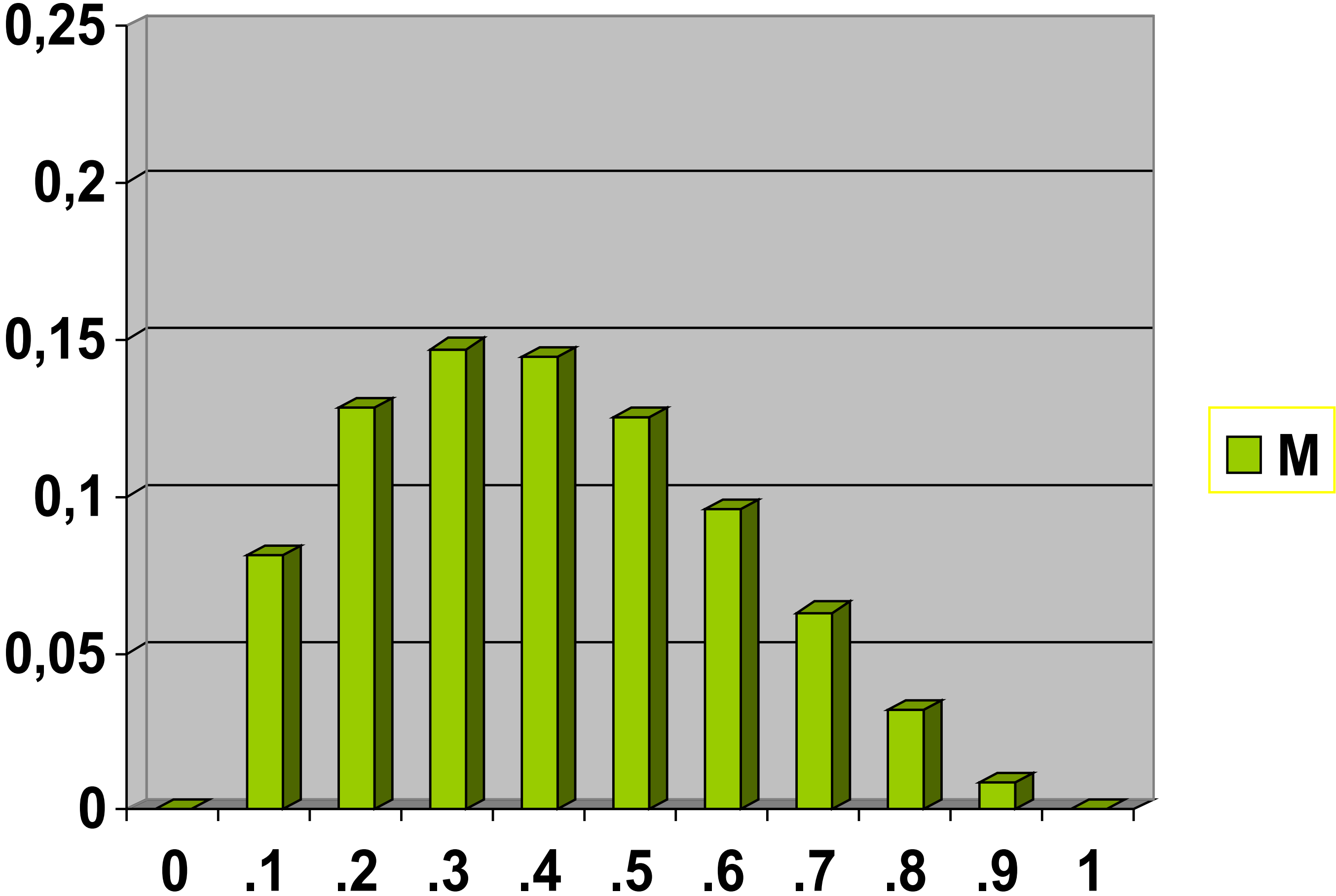
$$P(M(0.6)|D) = \frac{P(D|M(0.6))\,P(M(0.6))}{P(D)}$$

Calculating model likelihoods

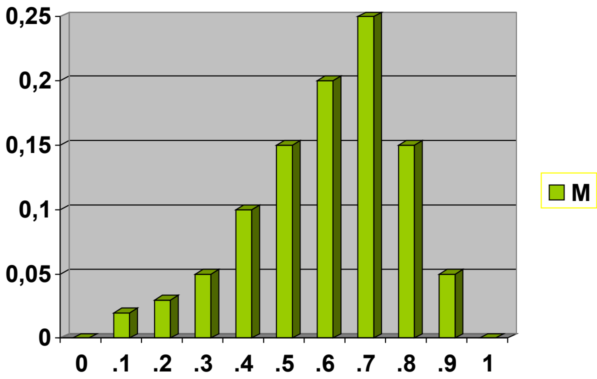
- i.i.d.: we assume that the observations are independent given any particular model M(Θ)
- P(ABB | M(0.6)) = 0.6 * 0.4 * 0.4 = 0.096
- This is repeated for each model M(Θ)

To calculate the *likelihood* of a model, multiply the probabilities of the individual observations given the model

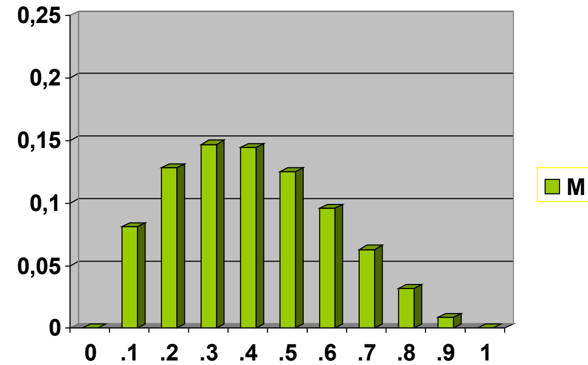
Likelihood histogram P(ABB|M(Θ))



Posterior = likelihood x prior



X



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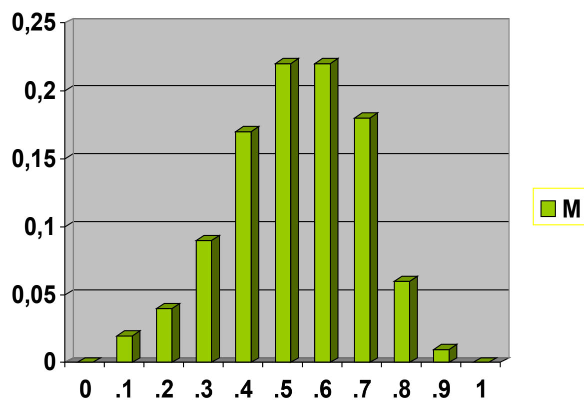


$$P\left(M\left(\theta\right)\middle|D\right)\propto P\left(D\middle|M\left(\theta\right)\right)P\left(M\left(\theta\right)\right)$$

The normalizing factor P(D)

$$P(M(\theta)|D) = \frac{P(D|M(\theta))\,P(M(\theta))}{P(D)}$$

Calculate:

$$P(D|M(0.0))\,P(M(0.0)) = s_1$$

$$P(D|M(0.1))\,P(M(0.1)) = s_2$$

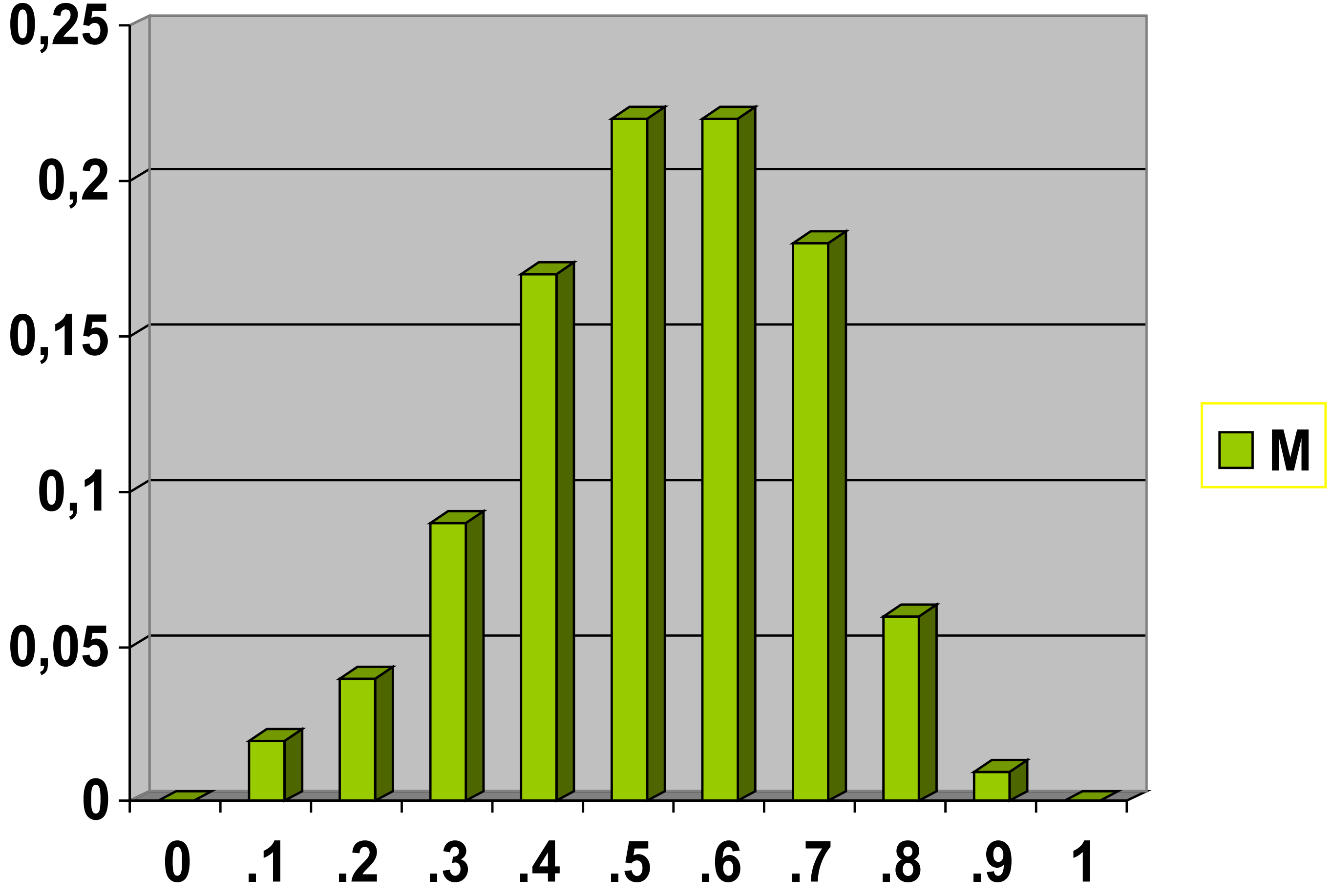
$$\ldots$$

$$P(D|M(1.0))\,P(M(1.0)) = s_{11}$$

Then:

$$P(D) = s_1 + s_2 + \ldots + s_{11}$$

Posterior distribution P(M(ϴ)|D)



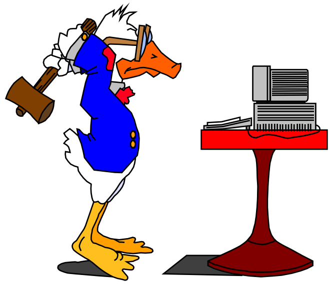
Predictive probability with data D

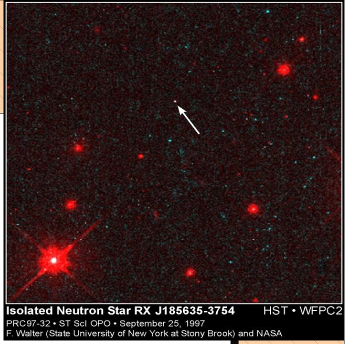
- With data D, the prediction is based on averaging over the models M(Ө) weighted now by the posterior (instead of the prior used earlier) probability of the models:

$$P(X \mid D) = \sum_i P(X \mid M_i, \cancel{D})\, P(M_i \mid D)$$

How did the probabilities change?

- The predictive probability P(A | D) = P(A|ABB) that the next (fourth) WLAN-card is OK came down from the prior 60% to 52% (the change is not great because the data set is small)



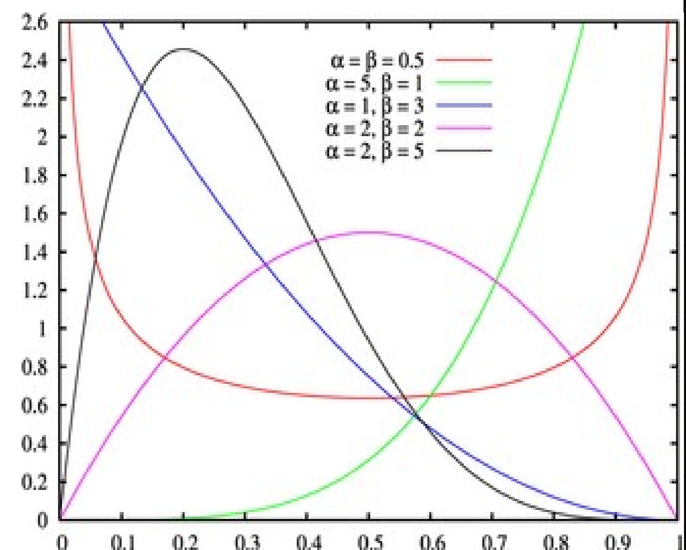


Densities for proportions

- a richer set of models allows more precise proportion estimates, but comes with a cost: the amount of calculations necessary increase proportionally
- we can move to consider infinite number of models
  - each model Θ is now a point on the interval from [0,1]
  - we get a “smoothed” bar chart called a density $P(\Theta)$
  - $\int P(\Theta)d\Theta=1$
  - only collections of models can have a probability > 0

Bayesian inference with densities?

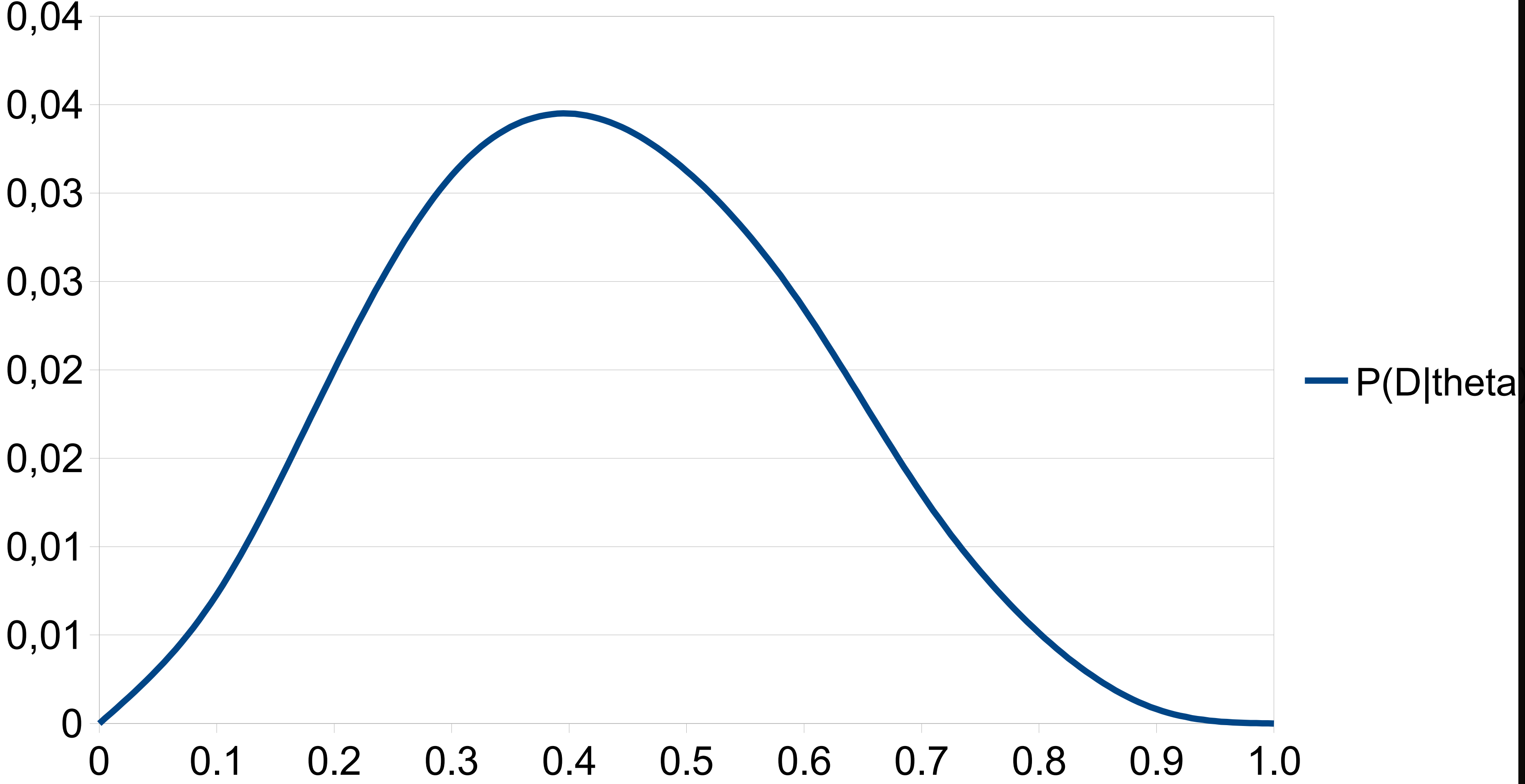
- Using densities means that we no longer add probabilities, but calculate areas
- To represent "infinite bar charts" we use curves that approximate the heights of bars
- But how to predict with densities? We cannot go over all the individual models as we did in the discrete case
- What about the prior?



Maximum likelihood

- Given a data D, different values of Θ yield different probabilities P(D|Θ). The parameters that yield the largest probability of P(D|Θ) are called maximum likelihood parameters for the data D.
  - $P(b,b,w,w,w|\Theta=0.7) = 0.7^2 0.3^3 = 0.1323$
  - $P(b,b,w,w,w|\Theta=0.1) = 0.1^2 0.9^3 = 0.00729$
  - $\mathrm{argmax}_\Theta\, P(b,b,w,w,w|\Theta) = \mathrm{argmax}_\Theta\, \Theta^2(1-\Theta)^3 = ?$

Likelihood P(b,b,w,w,w|Θ)



- NB! Not a distribution, but a function of Θ.

ML-parameters for the Bernoulli model.
(High school math refresher)

- So let us find ML-parameters for the Bernoulli model for the data with N_b black balls and N_w white ones.

$$P(D|\theta)=\theta^{N_b}(1-\theta)^{N_w},$$

so let us check when $P'(D|\theta)=0, \theta\in]0,1[$.

$$P'(D|\theta)=N_b\theta^{N_b-1}(1-\theta)^{N_w}+\theta^{N_b}N_w(1-\theta)^{N_w-1}\cdot-1$$

$$=\theta^{N_b-1}(1-\theta)^{N_w-1}[N_b(1-\theta)-\theta N_w]$$

$$=\theta^{N_b-1}(1-\theta)^{N_w-1}[N_b-(N_b+N_w)\theta]=0$$

$$\Leftrightarrow N_b-(N_b+N_w)\theta=0 \Leftrightarrow \theta=\frac{N_b}{N_b+N_w}$$

But ML-parameters are too gullible

- Assume D=(w,w), i.e., two white balls.
  - ML-parameter is $\Theta=0$.
  - Now P(next ball is black | $\Theta=0$)= 0.
  - Selecting ML parameters do not appear to be a rational choice.
- Be Bayesian:
  - Parameters are exactly the things you do not know for sure, so they have a (prior and posterior) distribution.
  - **Posterior distribution of the model is the goal of the Bayesian data-analysis.**

Predicting with posterior distribution

- Not a two phase process like in ML-case
  - first find ML parameters Θ.
  - then use them to calculate P(d|Θ).
- Instead: $$P(d|D) = \int_{\theta \in \Theta} P(\theta, d|D)$$
$$= \int_{\theta \in \Theta} P(d|\theta, D) P(\theta|D)$$
$$= \int_{\theta \in \Theta} P(d|\theta) P(\theta|D)$$
- Bayesian prediction uses predictions P(d|Θ) from all the models Θ, and weighs them by the posterior probability P(Θ|D) of the models.

Posterior for Bernoulli parameter

- So likelihood P(D|Θ) we can calculate.
- How about the prior P(Θ)?
  - We should give a real number for each Θ.
    - One way out: as earlier, use a discrete set of parameters instead of continuous Θ. (Works, is flexible, but does not scale up well.)
    - Another way: Study calculus.
- And how about $P(D)=\int_0^1 P(\theta)P(D|\theta)\,d\theta$

Prior for Bernoulli model

- The form of the likelihood gives us a hint for a comfortable prior
  - $P(D|\Theta) = \Theta^{Nb} (1-\Theta)^{Nw}$
  - If we define the $P(\Theta) = c\, \Theta^{\alpha-1} (1-\Theta)^{\beta-1}$,
    - c taking care that $\int P(\Theta)d\Theta = 1$, then
  - $P(\Theta)P(D|\Theta) = c\ \Theta^{Nb+\alpha-1} (1-\Theta)^{Nw+\beta-1}$
- Thus updating from prior to posterior is easy: just use the formula for the prior, and update exponents α-1 and β-1 (*conjugate* prior).

$P(\Theta)$ of a form $c\,\Theta^{\alpha-1}(1-\Theta)^{\beta-1}$ is called Beta(α,β) distribution

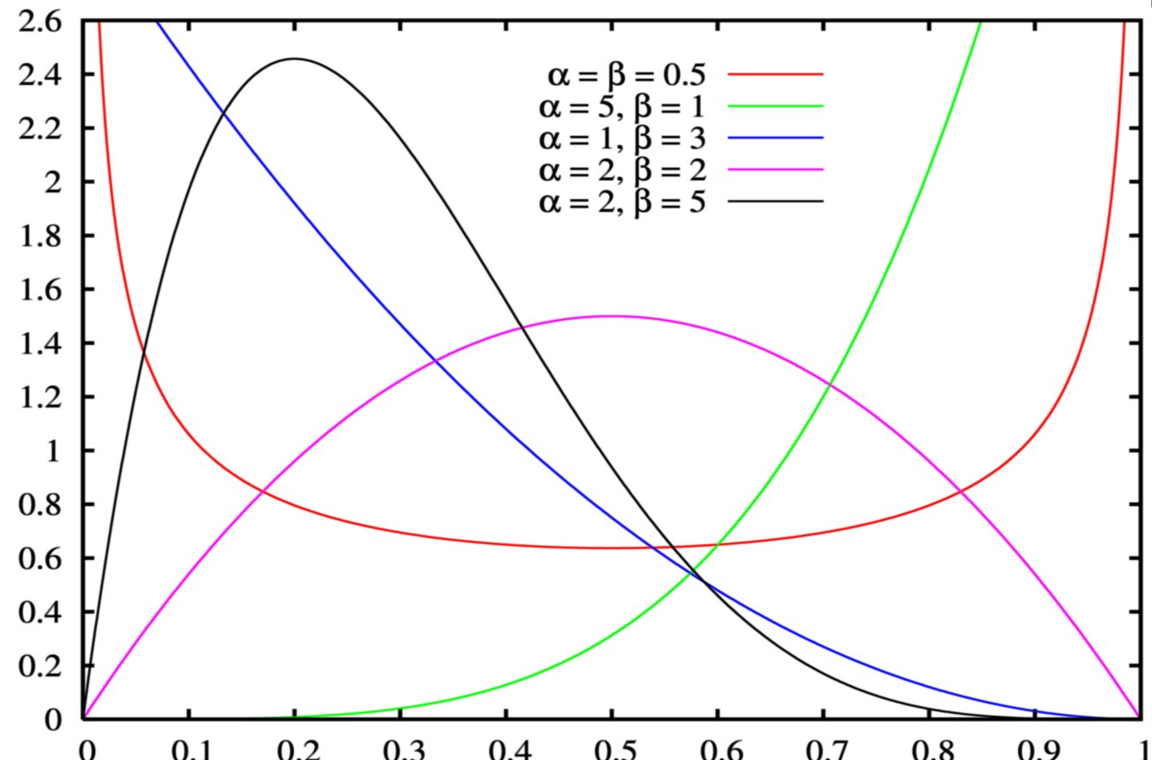
- The expected value of Θ is $\alpha/(\alpha+\beta)$.
- The normalizing constant is

$$c=\frac{1}{\int_0^1 \theta^{\alpha-1}(1-\theta)^{\beta-1}\,d\theta}$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)},$$

where Γ is the gamma function, a continuous version of the factorial:

$$\Gamma(n)=(n-1)!$$



Posterior of the Bernoulli model

$$P(\theta|D,\alpha,\beta)=\frac{\Gamma(\alpha+N_b+\beta+N_w)}{\Gamma(\alpha+N_b)\Gamma(\beta+N_w)}\theta^{\alpha+N_b-1}(1-\theta)^{\beta+N_w-1}$$

- Thus, a posteriori, Θ is distributed by Beta($\alpha+N_b,\beta+N_w$).
- And prediction:

$$\begin{aligned}P(b|D,\alpha,\beta)&=\int_0^1 P(b|\theta,D,\alpha,\beta)P(\theta|D,\alpha,\beta)d\theta\\&=\int_0^1 P(b|\theta)P(\theta|D,\alpha,\beta)d\theta=\int_0^1 \theta P(\theta|D,\alpha,\beta)d\theta\\&=E_P(\theta)=\frac{\alpha+N_b}{\alpha+N_b+\beta+N_w}.\end{aligned}$$

Bernoulli prediction

$$P(b|D,\alpha,\beta)=\frac{\alpha+N_b}{\alpha+N_b+\beta+N_w}.$$

- So P(b|w,w,α=1,β=1) = (1+0) / (1+0+1+2) = 1/4.
  - Sounds more rational!
  - Notice how the *hyperparameters* α and β act like extra counts.
  - That's why α + β is often called "equivalent sample size". The prior acts like seeing α black balls and β white balls before seeing data.

Laplace smoothing = Beta(1,1)

- For Bayesian inference, we can use a single model Θ^* which is the mean of the Beta(α,β) density:
  - $\Theta^* = (\alpha + N_+)/(\alpha + N_+ + \beta + N_-)$
- E.g.: flip a coin 10 times, observe 7 heads ("success"). Assuming a uniform prior Beta(1,1), the posterior for the Θ becomes Beta(8,4), and hence the predictive probability of heads is 8/12=2/3, or:
  - $\Theta^* = (7+1)/(10+2)$
- Also known as *Laplace's rule of succession* or *Laplace smoothing*

Equivalent sample size

- Predictive probabilities change less radically when $\alpha+\beta$ is large
- Interpretation: before formulating the prior, one has experience of previous observations - thus with $\alpha+\beta$ one can indicate confidence measured in observations
- Called “prior sample size” or “equivalent sample size”
- Beta(1,1) is the uniform prior
- Beta(0.5,0.5) is the *Jeffreys prior*

One variable, more than two values

- Variable X with possible values 1,2,...,n.
- Parameter vector $=(\Theta_1, \Theta_2, ..., \Theta_n)$ with $\sum_{n} \Theta_i = 1$.
- $P(X=i|\Theta)=\Theta_i$. Prior $P(\Theta)$ =

  Dirichlet$(\Theta; \alpha_1, \alpha_2, ..., \alpha_n)$ = $$\frac{\Gamma\left(\sum_{i=1}^{n} \alpha_i\right)}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \prod_{i=1}^{n} \theta_i^{\alpha_i - 1}$$
- Posterior $P(\Theta)=Dir(\Theta; \alpha_1+N_1, \alpha_2+N_2, ..., \alpha_n+N_n)$
- Prediction $P(x_i \mid D, \boldsymbol{\alpha})$ = $$\frac{\alpha_i + N_i}{\sum_{j=1}^{n} \alpha_j + N_j}.$$