Exercise 1:
Denote by $U = (u_1, \ldots, u_m)$ the first $m \leq p$ principal component directions (weights) of the zero mean random variable $x \in \mathbb{R}^p$. Assume you have $n$ observations of $x$, organized in the matrix $X \in \mathbb{R}^{p\times n}$.

1.1 Explain why the sample covariance can be expressed as $\frac{1}{n}XX^T$.

1.2 Let $z = U^T x$ and $Z = U^T X$. Give the horizontal rows of $Z$ an interpretation in terms of PCA.

1.3 Show that the rows of $Z$ are orthogonal to each other, and give an interpretation for this.

Exercise 2:

2.1 Assume the covariance matrix $C$ of a random vector $x \in \mathbb{R}^2$ has the form

$$C = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Calculate the eigenvalues as function of $\rho$ (by hand). What is the effect of correlation between the random variables on the eigenvalues?

2.2 Let $x_2 = ax_1 + \varepsilon$ so that

$$x = \begin{pmatrix} x_1 \\ ax_1 + \varepsilon \end{pmatrix}$$

where $a$ is some deterministic constant, and $\varepsilon$ is a random variable which is uncorrelated with $x_1$. Also assume that $E(x) = 0$. How do you have to choose the factor $a$ and the noise $\varepsilon$ so that $x$ has covariance matrix $C$?

2.3 Calculate the variance of $\varepsilon$ (the noise variance) and make a scatter plot of $x$ for $\rho \in \{-1, -0.25, 0, 0.5, 1\}$ (either sketch by hand or make the plots with matlab/R).

2.4 Suppose $X \in \mathbb{R}^{2\times n}$ is a sample of $n$ observations of $x$. What happens to the horizontal vectors $X_{1*}$ and $X_{2*}$ when $|\rho| \to 1$?

Exercise 3:
The objective function for a single factor is

$$J_{ls}(a) = \|C - aa^T\|^2,$$

where $C$ is the covariance matrix, and the squared norm $\|A\|^2$ of a matrix $A$ is defined as $\|A\|^2 = \sum_{ij} A_{ij}^2$. 

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3.1 Show that $\sum_{ij} A_{ij} B_{ij} = \text{Tr}(A^T B) = \text{Tr}(AB^T)$.

3.2 Write down the details of how to get from Equation (3) to

$$J_{ls} = \|a\|^4 - 2a^TCa + \text{Tr}(CC^T)$$ (4)

3.3 Calculate the gradient of $J_{ls}$, as defined in Equation (4), with respect to $a$.

3.4 Show that if the gradient is zero for some vector $v$, then $v$ is an eigenvector.

Let $e$ be an eigenvector of $C$ with unit norm. Find all the scalars $\alpha$ such that the vector $a^* = \alpha e$ makes the gradient of $J_{ls}$ zero.

3.5 Calculate the value of $J_{ls}$ for $a^*$. Conclude that the eigenvector $e$ that has the eigenvalue with the largest norm minimizes $J_{ls}$.

**Exercise 4:**

The optimization problem for quartimax is to maximize

$$J(U) = \sum_{ij} G((AU)_{ij})$$ (5)

under the constraint of orthogonality of $U$ (see Section 5.5).

4.1 Derive an iterative update rule for the matrix $U$ which solves the optimization problem for $G(y) = y^4$.

4.2 Show that $J(U)$ is constant for all orthogonal $U$ if $G(y) = y^2$. 