Computing Smallest MUSes of Quantified Boolean Formulas

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Motivation: Minimal Unsatisfiable Subsets (MUSes)

- Assuming monotonicity: **minimal explanations** as minimal sets of formulas S implying a consequence p
  - Relation to inconsistency: $S \rightarrow p$ is satisfiable iff $S \wedge \neg p$ is unsatisfiable
- Propositional logic: variety of **algorithms for computing minimal unsatisfiable subsets** (MUSes) (Marques-Silva & Mencía, 2020)
  - algorithms for smallest MUSes (Liffiton et al., 2009; Ignatiev et al., 2016, 2015)
    - corresponding decision problem Σ_2^p-complete (Liberatore, 2005)
  - MUSes of quantified Boolean formulas (QBFs) (Lonsing & Egly, 2015)

Computational complexity of and practical algorithms for computing smallest MUSes of QBFs remain unexplored.

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Motivation: Strong Explanations

- Nonmonotonic logics: if S implies p then $S' \supset S$ might not imply p
  - e.g. abstract argumentation: if a is credulously accepted in $F = (A, R)$, then a is rejected in $(A \cup \{d\}, R \cup \{d \rightarrow a\})$ for $d \notin A$
- **Strong explanations**: generalization to **nonmonotonic reasoning**
  - based on **strong inconsistency**: require that $S \subseteq K$ remains inconsistent for each S' with $S \subseteq S' \subseteq K$ (Brewka et al., 2019)
  - Reiter's **hitting set duality** satisfied: (minimal) explanations as hitting sets of (minimal) diagnoses and vice versa
  - instantiations: answer set programming, **abstract argumentation**

(Brewka & Ulbricht, 2019; Mencía & Marques-Silva, 2020; Ulbricht & Wallner, 2021)

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Motivation: Abstract Argumentation

- Strong explanations as induced subgraphs: acceptance status of argument unchanged no matter which arguments from original argumentation framework are added (Ulbricht & Wallner, 2021)
  - subset-minimality or smallest cardinality desirable
- **Declarative approaches** to extracting smallest strong explanations for credulous rejection: Σ_2^p-complete problem
  - answer set programming (Saribatur et al., 2020)
  - propositional SMUS extractors (Niskanen & Järvisalo, 2020)
- **Strong explanations for credulous acceptance** under admissible and stable semantics
  - verification of a strong explanation is Π_2^p-complete (Ulbricht & Wallner, 2021)
  - computing smallest explanations is (clearly) in Σ_3^p

Algorithmic approaches to computing strong explanations for credulous acceptance have not been investigated.

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Contributions

1. **Encodings** for computing strong explanations in abstract argumentation
   - smallest strong explanations as smallest MUSes of QBFs
   - for credulous acceptance and skeptical rejection
2. **Complexity** of computing smallest MUSes of QBFs
   - for k-QBFs: Σ^p_{k+1}-complete (leading $\exists$) or Σ^p_k-complete (leading $\forall$)
3. **Algorithm** for computing smallest MUSes of QBFs
   - based on the implicit hitting set (IHS) approach
   - employs modern QBF solving techniques
4. **Implementation** of the algorithm
   - generic: allows for computing smallest MUSes of QBFs in prenex CNF
   - empirical evaluation: practical declarative approach for computing strong explanations in abstract argumentation

Implementation and benchmark data openly available:
`https://bitbucket.org/coreo-group/qbf-smuser`

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Quantified Boolean Formulas (QBFs)

Boolean satisfiability (SAT) with quantifiers: $\exists$, $\forall$

- Instance: $\Phi = \overrightarrow{Q}_k.\varphi$
  - prefix $\overrightarrow{Q}_k = Q_1 X_1 \cdots Q_k X_k$:
    - alternating quantifiers $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$
    - pairwise disjoint sets of variables X_i
  - matrix φ: formula over $X_1 \cup \cdots \cup X_k$

Semantics defined recursively in a natural way:

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$S^\star \subset S$ is a **core** of $\exists S \overrightarrow{Q}_k . \varphi$
if $\exists S \overrightarrow{Q}_k . \underbrace{\varphi[S^\star]}_{=\varphi[s \mapsto \top \mid s \in S^*]}$ is false.

MUSes = subset-minimal cores
Smallest MUSes = smallest-cardinality cores

Note: definition covers all conjunctive forms, e.g. standard clausal MUSes.

Example

$$\exists x_1 y_1 z_1 \forall x_2 y_2 z_2$$
$$((x_1 \vee \neg x_2) \wedge (y_1 \vee \neg y_2) \wedge (z_1 \vee \neg z_2)$$
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Strong Explanations in Abstract Argumentation

- Argumentation frameworks (AFs):
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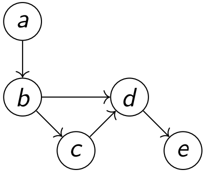
- Semantics σ characterize jointly acceptable sets of arguments $\sigma(F)$ called extensions
  - admissible, stable, ...
- Credulous acceptance: argument $q \in A$ contained in an extension $E \in \sigma(F)$
  - *In paper: skeptical rejection*



Strong explanations $S \subseteq A$: $q \in A$ remains credulously accepted in any subframework $F[A'] = (A', R \cap (A' \times A'))$ with $S \subseteq A' \subseteq A$.

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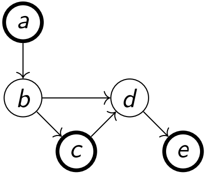

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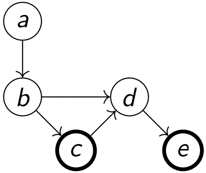

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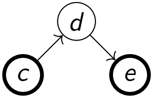

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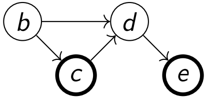

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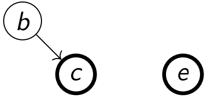

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(Ulbricht & Wallner, 2021)

Verification of a strong explanation hard for the second level!

Strong Explanations in Abstract Argumentation

- Argumentation frameworks (AFs): directed graphs $F = (A, R)$ (Dung, 1995)
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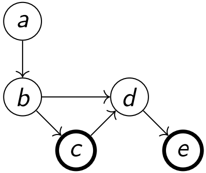

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Encodings for Strong Explanations in AFs

Variables:

- $Y = \{y_a \mid a \in A\}$: "argument a exists in subframework"
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$\rightarrow$ prefix $\exists Y \forall X$: "$\exists$ subframework $\forall$ extensions"

Propositional formulas $\varphi_\sigma(F)$ condition standard encodings of semantics σ on the existence of arguments: (Besnard & Doutre, 2004; Niskanen & Järvisalo, 2020)

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Strong explanations as cores of 2-QBFs

Let $S \subseteq A$. Now $Y[S] = \{y_a \mid a \in S\}$ is a core of $\exists Y \forall X \neg(\varphi_\sigma(F) \wedge x_q)$ iff S is a strong explanation for credulously accepting q in F.

Smallest strong explanations as smallest MUSes of a 2-QBF!

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Complexity of Computing Smallest MUSes of QBFs

How hard is it to decide whether a k-QBF admits a small core?

Leading quantifier $\exists$: Σ^p_{k+1}-complete.

- Generalizes Σ^p_2-completeness for propositional logic. (Liberatore, 2005)
- Problem remains hard for DNF formulas when k is odd...
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Implicit Hitting Set (IHS) Approach

Identify an increasing collection of *non-solutions*
and *exclude them from consideration* in a minimal way.

(Moreno-Centeno & Karp, 2013; Saikko et al., 2016)

- A *correction set* $cs \subseteq S$ renders $\exists S \overrightarrow{Q}_k . \varphi[S \setminus cs]$ true.
  - QBF solver: *extract a collection of correction sets* $\mathcal{C}$.
- hs is a *hitting set* over $\mathcal{C}$ if hs intersects each $cs \in \mathcal{C}$.
  - IP solver: compute *hitting sets with smallest cardinality*.

Reasoning and *optimization* effectively decoupled:

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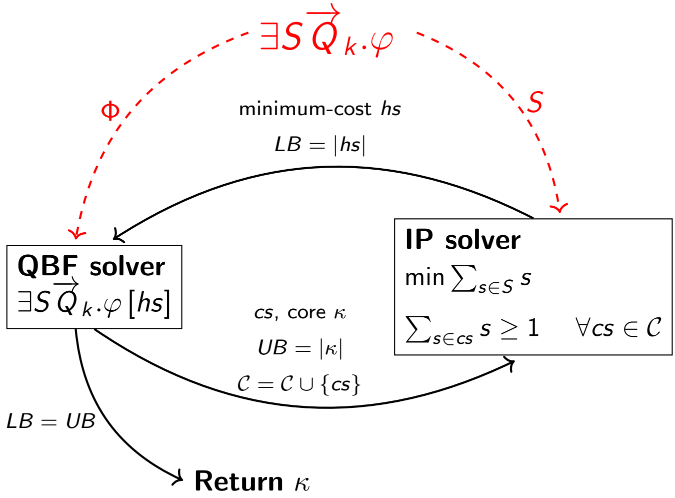

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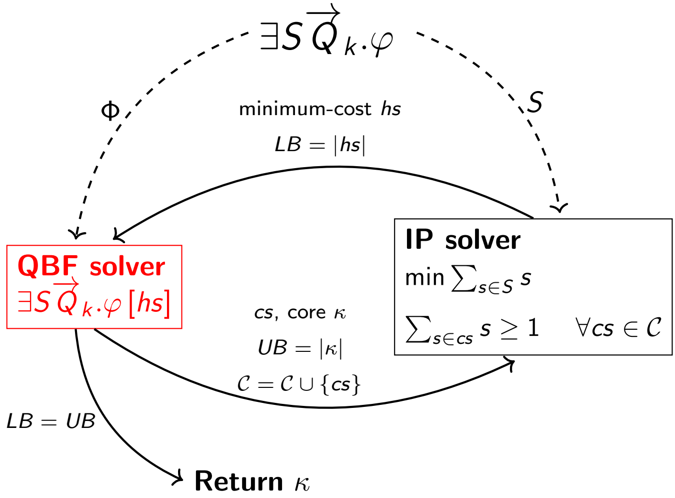

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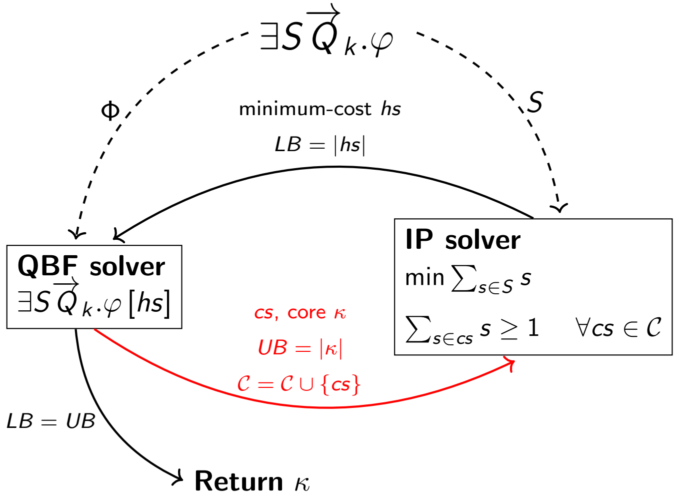

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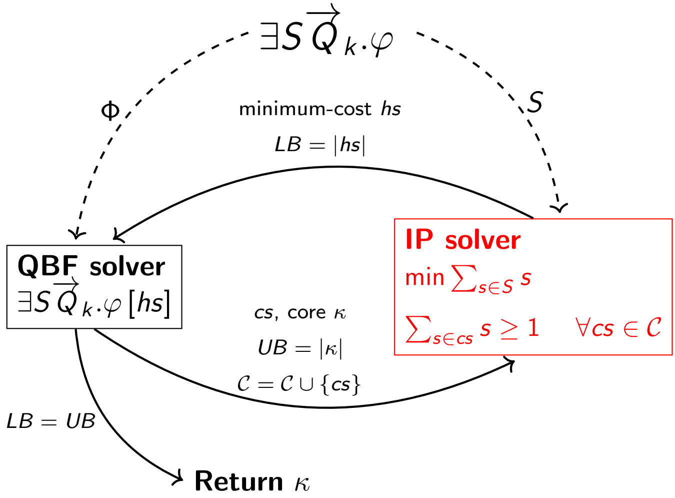

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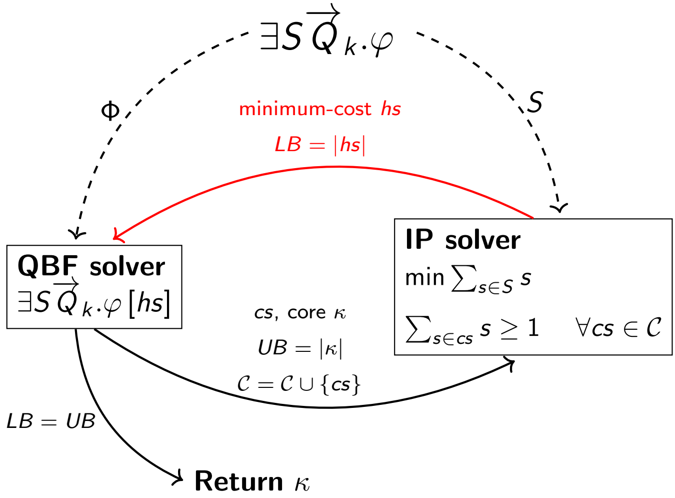

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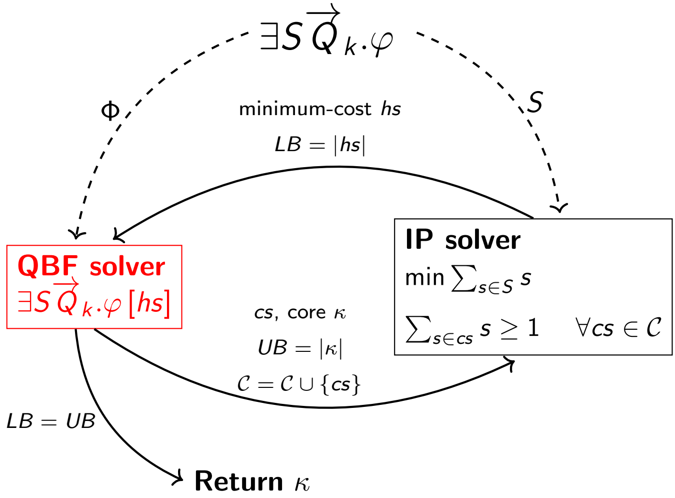

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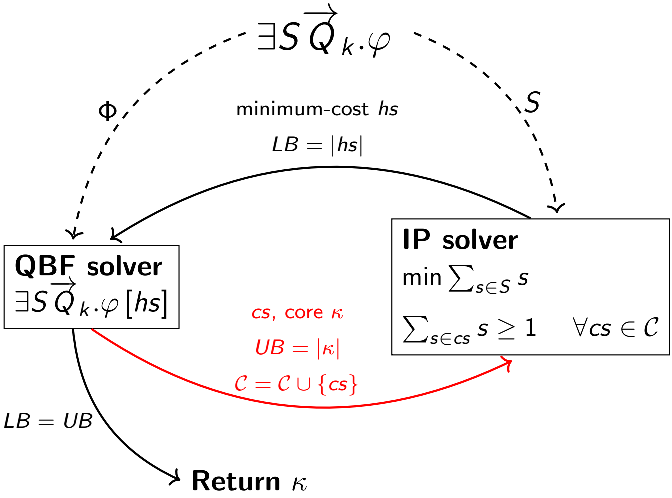

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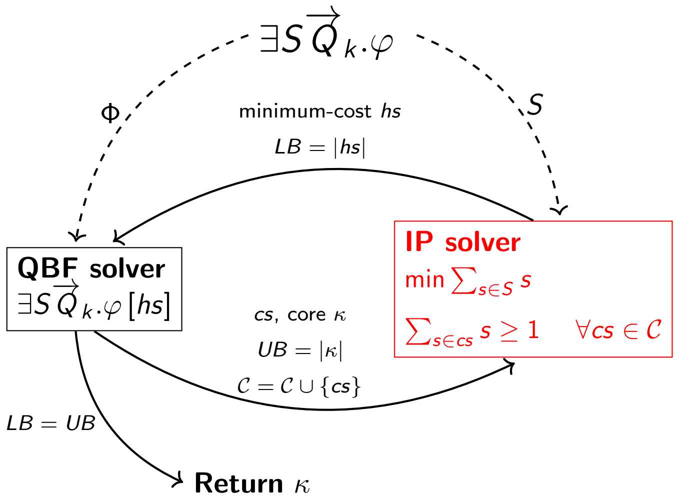

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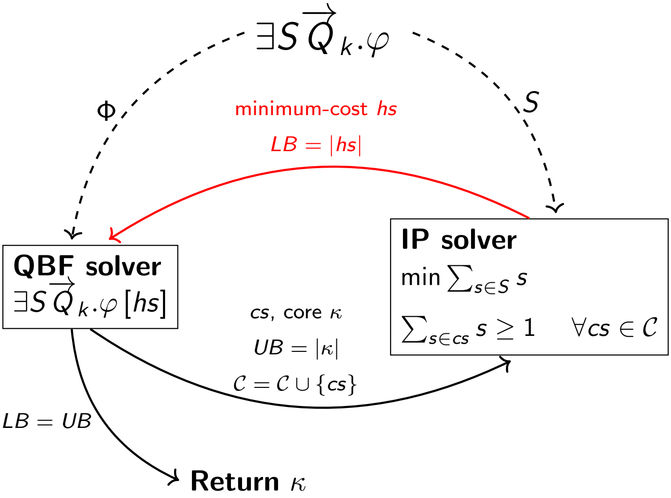

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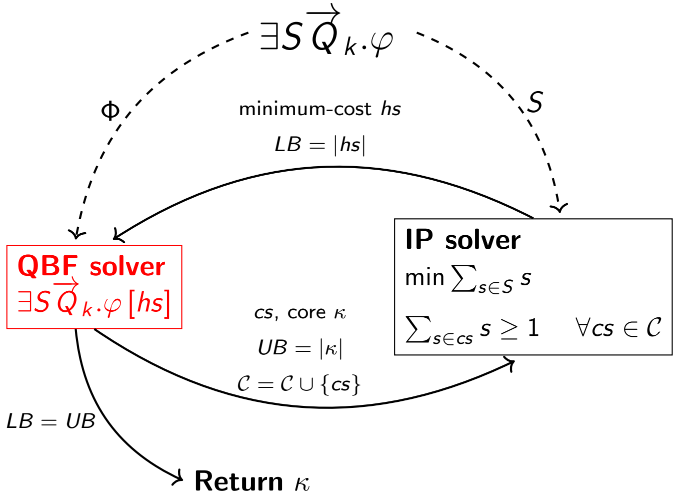

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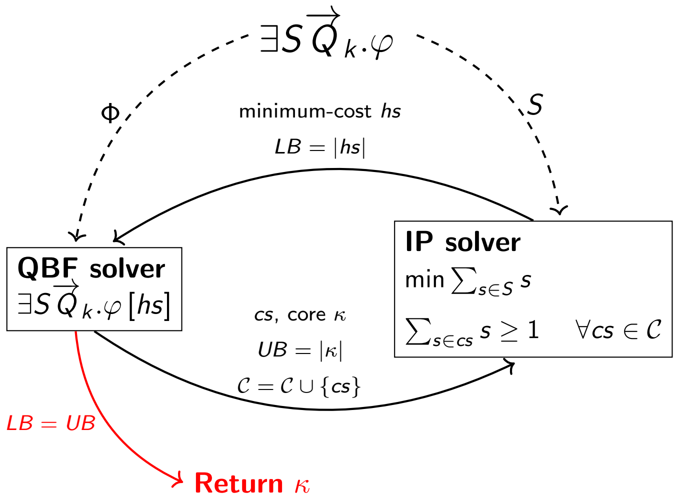

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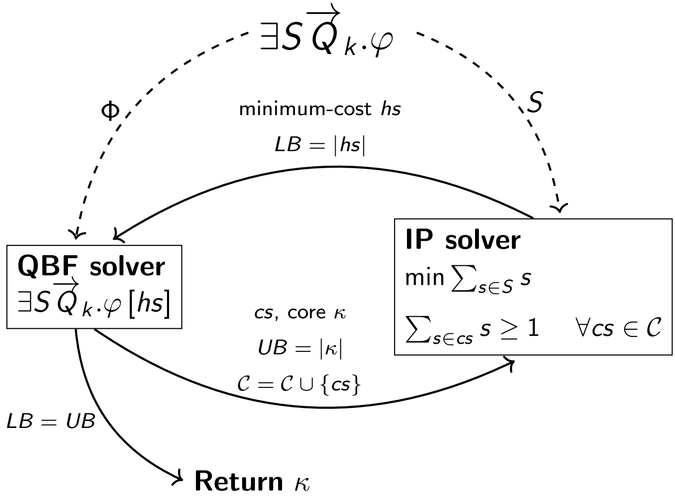

Empirical Evaluation

Benchmark Instances

- Strong explanations: all 326 AFs from ICCMA'19
  - semantics: admissible, stable
  - query arguments sampled from credulously accepted arguments
- *In paper: specific small unsatisfiable QBFLIB instances!*

Benchmark setup

- QBF solvers: RAReQS, DepQBF
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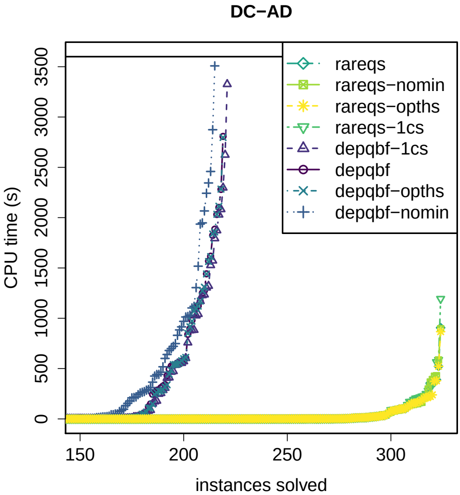

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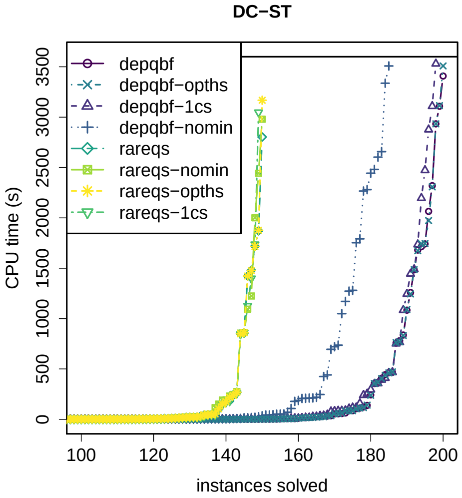

Summary

Contributions

A first overview on the computation of smallest MUSes of QBFs:

- Computational **complexity analysis**
  - Σ_{k+1}^{p}-complete for leading existential quantifier
  - Σ_{k}^{p}-complete for leading universal quantifier
- IHS-based **algorithm and implementation**
  - relies on iterative QBF and IP solver calls
  - additional techniques can be incorporated
- Application: **declarative encodings** for computing smallest strong explanations in abstract argumentation
  - empirical evaluation shows that the approach is viable

Implementation available online in open source:
`https://bitbucket.org/coreo-group/qbf-smuser`

Thank you for your attention!

Get in touch via email:

`andreas.niskanen@helsinki.fi`

Or come chat in person :)

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Extra: Clausal MUSes as Cores

Let $\varphi_{CNF} = \{C_j \mid j = 1, \ldots, m\} = \bigwedge_{j=1}^{m} C_j$ be a CNF formula.

The MUSes of $\Phi_{PCNF} = \overrightarrow{Q}_k.\varphi_{\mathrm{CNF}}$ correspond exactly to subset-minimal cores of

$$\exists S \overrightarrow{Q}_k. \bigwedge_{j=1}^{m} (s_j \rightarrow C_j)$$

with $S = \{s_1, \ldots, s_m\}$: if $S^* \subset S$ is a core, then $\overrightarrow{Q}_k. \bigwedge_{j:\ s_j \in S^*} C_j$ is false, and vice versa.

Same holds for smallest MUSes and smallest-cardinality cores.