8 Modeling network traffic using game theory

Network represented as a **weighted graph**; each edge has a designated travel time that may depend on the amount of traffic it contains (some edges sensitive to congestion and some not). In addition, for example all assumed to travel from A to B.

![Network Diagram](image)

See figure 8.1

The network is **crowded** => we should make an evaluation of the available routes before heading off.

If all cars go through C, then total travel time for all is $4000/100+45 = 85$;

If cars divide up evenly, then travel time for all cars is $2000/100+45 = 65$.

### 8.1 Equilibrium traffic

In the real-world games, there are an enormous number of players and also a huge set of strategies. The payoff to each player depends on the strategies chosen by all players.

A **Nash equilibrium** is a list of strategies, one for each player, so that each player's strategy is a best response to all the others. **No benefit from deviating** to another strategy.

(That is to say, “win for one, and win for all”.)

For this transportation network example, **Nash equilibrium** exists when the **drivers balance**
themselves evenly between the two routes (no reason to deviate).

8.2 Braess's Paradox

The idea is that adding a new strategy (a resource) to a game makes things worse (travel times become longer) for everyone instead in equilibrium. Informal sense: by upgrading the network is a good thing. (So, when designing networks, we need to think about how to prevent bad equilibrium from arising beforehand.)

See figure 8.2

When an express has been established from C to D, choosing this route becomes a dominant strategy for all drivers: regardless of the current traffic pattern, you gain by switching your route to go through C and D because it is fast. \( 4000/100 < 45 \)

This is also a unique Nash equilibrium in this new highway network. However, it leads to slower traffic since every driver chooses this dominant strategy.

8.3 The social cost of traffic at equilibrium

(how far from optimal traffic can be at equilibrium)

Assume that network is any directed graph, traffic is denoted by \( x \) and each edge \( e \) has a travel time function \( T_e(x) = xa_e + b_e \) (linear in the amount of traffic \( x \)).
See figure 8.3

**Traffic pattern** = the choice of a path by each driver

**Social cost of a given traffic pattern** = the sum of travel times incurred by all drivers when they use this traffic pattern.

**Socially optimal traffic pattern** = the traffic pattern with the minimum possible social cost.
(there might be multiple traffic patterns for a certain network)

Assume four drivers in the situation of figure 8.3. Now **socially optimal traffic pattern costs 28**, while the traffic pattern of **Nash equilibrium costs 32**.

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![Diagram](image)

(a) *The social optimum.*

(b) *The Nash equilibrium.*

Figure 8.4: A version of Braess's Paradox: In the socially optimal traffic pattern (on the left), the social cost is 28, while in the unique Nash equilibrium (on the right), the social cost is 32.

See figure 8.4

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**A. How to find a traffic pattern at equilibrium**

**Best-response dynamics** =

- If no player can do better => equilibrium found
- If at least on player can change to a better strategy => reconfigure the player’s strategy to the better one
Best-response dynamics stops at equilibrium, where everyone is playing his/her best response to the current situation.

Following proves:
1. best-response dynamics must always terminate in an equilibrium.
2. equilibria can be reached by a simple process during which drivers constantly update what they're doing according to the best response.

8.A.1 Analyzing best-response dynamics via potential energy (of a traffic pattern)

To guarantee termination, a progress measure should be defined to track the process as it operates. Potential energy of a traffic pattern is used here as an alternate quantity which strictly decreases with each best-response update.

Potential energy of edge e = \( T_e(1) + T_e(2) + \ldots + T_e(n) \)

It is a cumulative quantity based on the assumption that drivers cross the edge one by one, and each driver only feels the delay caused by himself and the drivers who cross the edge in front of him.

Potential energy of a traffic pattern is the sum of the potential energies of all the edges with their current number of drivers in the traffic pattern.

8.A.2 Proving that best-response dynamics comes to an end

According to figure 8.5, as more and more drivers choose route C to D, the potential energy decreases, and social cost increases meanwhile, and finally equilibrium is achieved.

According to figure 8.6, when a driver switches to another route, two things happen
- abandon the current route (Abandoning the upper path releases 2 + 5 = 7 units of potential energy)
- adopt a new route (adopting the new zigzag path adds 2+0+3=5 units of potential energy back into the system.

Finally, the result is a decrease of 2 units in potential energy.

Proof of why the potential energy(PE) strictly decreases throughout best-response dynamics:

1. the PE of the current edge a driver is driving on is
   \( T_e(1) + T_e(2) + \ldots + T_e(x-1) + T_e(x) \)
2. When the driver **abandons** the current edge, its PE becomes
   \[ T_e(1) + T_e(2) + \ldots + T_e(x-1) \]

3. Hence **the change in PE** of the previous edge \( e \) is \( T_e(x) \), *exactly the travel time* that the driver was experiencing on \( e \).

4. **Before** the driver chooses the **new route** \( \hat{e} \), the new route's PE is
   \[ T_{\hat{e}}(1) + T_{\hat{e}}(1) + \ldots + T_{\hat{e}}(x) \]

5. After the driver **switch to** this **new route**, the new PE becomes
   \[ T_{\hat{e}}(1) + T_{\hat{e}}(1) + \ldots + T_{\hat{e}}(x)+ T_{\hat{e}}(x+1) \]

6. So this switch behavior causes an **increase of** \( T_{\hat{e}}(x+1) \) in PE, *exactly the new travel time* the driver experiences on this new edge.

7. From this, we get that:
   \[ \text{the net change in PE} = \text{new travel time} - \text{old travel time} = T_{\hat{e}}(x+1) - T_e(x) \]

8. Since the driver **only changes** his path when it **speeds up** (best-response dynamics)
   => new travel time < old travel time
   => **the net change in PE is negative** for any best-response action.

9. To conclude, we have proven that the potential energy **strictly decreases** throughout best-response dynamics.

**8.8 B The relationship between potential energy and the total travel time:**

The potential energy of an edge \( e \) is:
\[ \text{Energy}(e) = T_e(1) + T_e(2) + \ldots + T_e(x) \]

since each driver experiences a travel time of \( T_e(x) \), the total travel time experienced by \( x \) drivers on edge \( e \) is:
\[ \text{Total-Travel-Time}(e) = x \times T_e(x) \]

based on the assumption that \( T_e(x) \) is **linear** which can be represented by \( T_e(x) = ax + b \), we get:
\[ 1/2 \times \text{Total-Travel-Time}(e) \leq \text{Energy}(e) \leq \text{Total-Travel-Time}(e) \]

from this, we know that PE is **between the total travel time** and **half the total time**.

**8.8 B.1 Relating the travel time at equilibrium and social optimality**
Let $Z$ be a traffic pattern

**Social-Cost($Z$)** is the sum of the total travel time on all edges.

Social-$\text{Cost}(Z) = \text{Sum over edges (Total-Travel-Time(e))}$

$1/2\times\text{Social-Cost}(Z) \leq \text{Energy}(Z) \leq \text{Social-Cost}(Z)$

Now we let $Z$ be the pattern where **social optimum** is achieved, and $\hat{Z}$ be the pattern where **equilibrium** is got.

Energy($\hat{Z}$) $\leq$ Energy($Z$)

Social-$\text{Cost}(\hat{Z}) \leq 2\times\text{Energy}(\hat{Z})$

and, $\text{Energy}(Z) \leq \text{Social-Cost}(Z)$

from these, we get:

**Social-$\text{Cost}(\hat{Z}) \leq 2\times\text{Energy}(\hat{Z}) \leq 2\times\text{Energy}(Z) \leq 2\times\text{Social-Cost}(Z)**

To conclude, the **potential energy decreases during the best-response dynamics**, preventing the social cost from every increasing by more than a factor of two. Thus, we can **use PE to put a bound on the social cost of the equilibrium that is reached**.

**9 Auctions**

eBay, short-term loans, commodities

**9.1 Types of auctions**

Assumptions

-A seller auctioning one item to a set of buyers
-Each buyer (bidder) has a **intrinsic/true value** (highest price for purchasing this item).

**Ascending**-bid auctions (English auctions)

**Descending**-bid auctions (Dutch auctions)

**First-price sealed**-bid auctions

**Second-price sealed**-bid auctions (Vickrey auctions)
9.2 When are auctions appropriate?

**Know values** the case where seller and buyers know each other’s valuations.

Assume that **seller values** item at x, and the **maximum value** held by a potential buyer is y which is bigger than x
- seller sets the price at a **fixed price just below** y
- the seller will get almost full value of surplus(y-x) by announcing a right price.

**Unknown values** refers to a setting where participants do not know each other’s valuations, buyers has **Independent private values**. In the following analysis we assume that bidders are in this case.

9.3 Relationships between different auction formats

Our **main goal** is to formulate auctions as game, and choose a appropriate strategy to get the highest payoff.

9.3.1 Descending-bid and First-Price auctions

**Descending-bid** : no bidder says anything until finally someone accepts the current price.

**Descending-bid** auction is **equal** to **First-Price** auction.
- descending-bid: for each bidder i, there is a first price b_i at which she is ready to accept.
- sealed-bid first-price: item goes to bidder with the highest bid and she pays the price that equals the bid

9.3.2 Ascending-bid and Second-Price auctions

In an **ascending-bid** auction the seller **increases** the price incrementally.

Bidder stays in an **ascending-bid** auction up to the price reaches her true value. The person with the highest bid stays longest and winning the item.

We can reformulate the **ascending-bid** : Winner in this auction pays the **price at which the second-last person dropped out**.

**Second-Price sealed-bid** auction is close similar to **Ascending-bid** auction.

**Bidding** one’s true valuation is a **dominant strategy** in **second-price sealed-bid** auction.

**First-Price auctions and Second-Price auctions**

Why not sellers only use the first-price auction which seems to get more money? 
Bidders in a **fist-price auction** will tend to **bid lower** than they do a **second-price auction**.
When two people submit the same bid, and it’s tied for the largest? 
In this case **first-price auctions** and **second-price auctions** are equal.

### 9.4 Second-price auctions

**Payoff** to bidder $i$ with value $v_i$ and bid $b_i$ **(sealed-bid second-price auction)**

$$
= 0, \text{ if } b_i \text{ is not the winning bid} \\
= v_i - b_k, \text{ if } b_i \text{ is the winning bid and } b_k \text{ second-place bid}
$$

### 9.4.1 Truthful bidding in second-price auctions

Assume **independent private valuations**

**Truthful bidding** $= \text{ bidding one’s own valuation}

**Claim:** In a sealed-bid second-price auction, it is a **dominant strategy** for each bidder $i$ to choose a bid $b_i = v_i$ (bid equals valuation)

---

Figure 9.1: If bidder $i$ deviates from a truthful bid in a second-price auction, the payoff is only affected if the change in bid changes the win/loss outcome.
Proof consists of two cases:
- \( b'^i_i > v_i \): affects payoff only if bidder i would lose with bid \( b'_i \) and win with \( b'^i_i \). This happens only if the second-place bid \( b_k \) is between \( b_i \) and \( b'_i \): 
  \( b_i < b_k < b'^i_i \), but this means that payoff \( v_i - b_k \leq 0 \).
- \( b''i_i < v_i \): affects payoff only if bidder would win with bid \( b_i \) and lose with bid \( b''_i \). Thus, by deviating, bidder loses (payoff = 0) instead of winning and making payoff \( v_i - b_k \geq 0 \).

\[ \Rightarrow \] In both cases smaller payoff when deviating, so bid truthfully.

Because truthful bidding is a dominant strategy, it is the best thing to do regardless of what the other bidders are doing.

In a second-price auction, the bid price of player i only affects if player i wins or loses, while the price player i finally pays (if she wins) is determined by the point at which the second-place bidder drops out.

9.5 First-price auctions and other formats

In a first-price auction, the bid price not only affects whether player i win or loose, but also how much player i finally pays (if she wins).

Payoff to bidder i with valuation \( v_i \) and bid \( b_i \)

\[
\begin{align*}
&= 0, \text{ if } b_i \text{ is not the winning bid} \\
&= v_i - b_i, \text{ if } b_i \text{ is the winning bid}
\end{align*}
\]

Truthful bidding no longer dominant strategy, because by bidding so payoff equals in all cases zero.

Optimal way is to shade bid slightly downward from the the valuation \( v_i \), and there is a trade-off between being the winner and paying too much and not winning .

- Need knowledge about other bidders’ and their distribution of valuations

9.5.1 All-pay auctions

For example political lobbying, preparing a competitive job application

Payoff to bidder i with valuation \( v_i \) and bid \( b_i \)

\[
\begin{align*}
&= -b_i, \text{ if } b_i \text{ is not the winning bid} \\
&= v_i - b_i, \text{ if } b_i \text{ is the winning bid}
\end{align*}
\]

Since everyone has to pay his/her bid price regardless of the auction result, the bids will typically shade much lower than in a first-price auction.

9.6 Common values and the winner’s curse
Assuming **independent valuations does not make sense** if the buyer is about to resell the object.

Assume that each bidder has some private knowledge about the valuation of the auctioned item.

Assume that each bidder’s estimate \( v_i \) of the true value is defined by
\[
v_i = v + x_i,
\]
where \( v \) is the true value and \( x_i \) is error in bidder i’s estimate (average(\( x_i \)) = 0).

Assume a **second-price sealed-bid auction** (truthful dominant strategy to bid own valuation)

=> bidding one’s valuation is not dominant strategy anymore

=> by winning, the winner learns that her estimate was the highest, which means

that it was also **more likely an over-estimate than an under-estimate**.

This is an example of **Winner’s curse**, and generally it **means** that it is expected that the bidders will shade their bids downward even when the second-price sealed-bid format is used (with first-price format even further).

When there is a large amount of bidders, anyone who in fact makes an error and overbids is more likely to be the winner of the auction.

**9.7 Bidding strategies in first-price and all-pay auctions**

Idea is to develop models under which one can derive equilibrium bidding strategies in the auctions presented earlier.

In general, valuations are assumed to be independently and uniformly distributed on interval from 0 to 1.

In the following, we will explore how optimal behavior varies depending on the number of bidders and distribution of values.

**9.7.A Equilibrium bidding in first-price auctions**

Assumption: the bidders know in advance about the amount of competitors and partial information about their competitors’ values for the item.

A **strategy** for a bidder is a function \( s(v) = b \) that maps her true value \( v \) to a non-negative bid \( b \).

- function \( s \) is a strictly increasing and differentiable (**different values produce different bids**)  
- \( s(v) \leq v \) for all \( v \) (**never bid above valuation**)

**9.7.A.1 Two bidders: the relevance principle**
Bidder 1 has valuation \( v_1 \) and bidder 2 has valuation \( v_2 \)

\[ \Rightarrow \text{Bidder 1 wins with probability } P(v_1 > v_2) = v_1 \]

When bidder 1 wins, her payoff is \( v_1 - s(v_1) \), which means that her expected payoff is

\[ g(v_1) = v_1 - (v_1 - s(v_1)) \]

**Strategy** \( s \) is an **equilibrium strategy**

if for each bidder \( i \) there is no incentive to deviate from strategy \( s \) when all the competitors are also using \( s \).

**Revelation principle**

What is the condition that player \( i \) does not want to deviate from strategy \( s(.) \)?

\[ v_i \ast (v_i - s(v_i)) \geq v \ast (v_i - s(v)) \text{ for all } v \text{ in } [0,1] \]

\( s(v) = v/2 \) satisfies this.

**Conclusion**

If two bidders share a same strategy of assigning bid price, then **equilibrium for each is to shade their bid down by half**. However, this is just an equilibrium for this scenario, in solving for a bidder’s optimal strategy, we used each bidder’s expectation about her competitor’s bidding strategy.

**9.7.A.2 Equilibrium with many bidders**

**Result:** \( s(v_i) = (n-1)/n \ast v_i \)

So, in order in get an equilibrium, each bidder should shade her bid down by a factor of \((n-1)/n\).

As the number of bidders increases, bidders have to bid more aggressively.

**9.7.B Seller Revenue**

**Check assumptions from page 265!**

In a **second-price auction**, the seller gets expected revenue of \( (n-1)/(n+1) \).

In a **first-price auction**, the seller gets expected revenue of \( (n-1)n \ast n/(n+1) = (n-1)/(n+1) \).

So these two types of auctions produce exactly the same expected revenue for the seller.

**9.7.B.2 Reserve Prices**

It is better for the seller to announce a reserve price of \( r \) before running the auction.

With a single bidder, the optimal reserve price is \( r = (1+u)/2 \), halfway between the value of the
object to the seller and the maximum possible bidder value.

**9.7.C Equilibrium Bidding in All-Pay Auctions**

Check yourself if interested.

**Exercises:**

chapter 8: 1.a, 1.b
chapter 8: 1.c
chapter 9: 1
chapter 9: 8