Kurt Gödel
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1. Foreword

This is a report on Kurt Gödel prepared for the University of Helsinki 2016 seminar on pioneers of computer science. Gödel was a logician, mathematician and a philosopher. He was the greatest logician of the 20th century, perhaps the greatest since Aristotle. Gödel wasn't a computer scientist, but his work was of fundamental importance to the birth of theoretical computer science. Gödel's work was especially important to the formulation and acceptance of the Church-Turing thesis that states that every effective computation can be carried out by a Turing machine. The second chapter of this paper gives a biographical sketch. It is divided in two parts: Gödel's life before the Second World War and his later life in the USA. The third chapter gives brief background and explanation of Gödel's most important results: his incompleteness theorems.

2. Biographical Sketch

2.1 Gödel in Vienna

Kurt Gödel was born in Brno, currently a city in Czech Republic but at the time part of Austria-Hungary, on 28 April 1906. He was the second and last child of Rudolf, a director of a textile factory, and Marianne Gödel. They were German speaking and nominally Christian. Kurt Gödel's family was reasonably affluent and his childhood was generally uneventful, although he did suffer from health problems for much of his childhood. He had a particularly warm relationship with his mother. He was a good pupil at school and only once received less than the highest mark in a subject. Ironically that subject was mathematics. (Dawson: 3-18.)

Kurt Gödel began his studies at the University of Vienna in the autumn of 1924. Gödel studied mainly physics for the first one or two years, but switched to mathematics after being impressed by the lectures of Philipp Furtwängler. He did also take multiple courses in philosophy. In 1926 he became a member of an informal group of intellectuals known as The Vienna Circle. He attended it's meeting regularly until 1928, but only seldom after that. He did, however, disagree with many of the Circle's views, particularly the idea that mathematics should be considered as syntax of language. (Dawson: 21-34.)
Gödel completed his doctoral thesis on the Completeness Theorem in 1929. Only a year earlier Hilbert and Ackerman had proposed the question about the completeness of the first-order logic. The Completeness Theorem basically states that within first-order logic every valid logical expression is provable without needing to use any additional rules, and the first-order logic is in that sense complete. In 1930 Gödel published a paper based on his thesis that includes the Compactness Theorem that would later become one of the main tools of model theory. (Kennedy 2015.)

Gödel mentioned the possibility that there might be formally undecidable statements within arithmetic in the introduction to his 1929 doctoral thesis. This was at the time a very surprising and unlikely idea, however in the paper published in 1931 Gödel showed that it was indeed so and presented his two famous incompleteness theorems. The First Incompleteness Theorem exhibits an arithmetic statement which is neither provable nor refutable in certain form of arithmetic. The Second Incompleteness Theorem shows that the consistency of arithmetic cannot be proved within arithmetic itself. (Kennedy 2015.)

In the early 1933 Gödel was appointed as Privatdozent. This was an essential step in entering academic career in the German university system, although it did not yet guarantee a good living. In the autumn of 1933 Gödel traveled to America. He had been invited to Princeton's newly formed Institute for Advanced Study as a visiting scholar. For him one big reason for going to Princeton seems to be the possibility to consult with Alonzo Church about his formalism. (Dawson: 87-101.)

Gödel traveled back to Europe, where the political situation at this time had begun to be rather restless and ominous, on May of 1934. In Austria Gödel's mental and physical health deteriorated. He was diagnosed as having suffered a nervous breakdown because of overwork and was treated at a sanatorium in Vienna. Gödel had been planning to go back to Princeton for the spring term of 1935, but because of his health problems this was postponed to autumn. He was however able to do some important mathematical work at this time. Of particular interest from the point of view of theoretical computer science is a presentation he gave that summer titled "On the Length of Proofs", which was published in 1936. Later, with the development of computational complexity theory, theorem presented there was recognized as an early example of a speed-up theorem. (Dawson: 103-108.) However, Gödel didn't give a proof for the theorem and the first full proof for Gödel's speed-up theorem was given only in 1994 (Buss 1994).
Gödel considered himself healthy enough to travel to Princeton for the autumn term of 1935. His work there was however interrupted by health problems, apparently depression, and he had to make an early return to Europe on late November. One sad consequence of this was that otherwise Gödel might have stayed in Princeton longer and met Alan Turing when he went there in 1936. As things happened, the two men never did meet. The year 1936 was difficult for Gödel and he spent several months being treated in a sanatorium for nervous diseases. He had an obsessive fear of being poisoned and he spend a lot of time reading about related things such as toxicology. He's fear of poisoning and dietary concerns would get even worse on later years. Gödel wasn't able to return to his academic work in Vienna properly before the summer of 1937. (Dawson: 108-113.)

Gödel had been seeing a girl named Adele for many years, but Gödel's parents strongly disapproved their relationship. Adele was more than six years older than Gödel, came from a lower-class family and, what was most suspicious, she was a dancer. Kurt and Adele finally married in September of 1938 in a private civil ceremony. (Dawson: 128-140.) Soon after the wedding Gödel again traveled to America without his wife. There Gödel lectured at the universities of Princeton and Notre Dame. He returned to Vienna in the summer of 1939. (Dawson: 34.)

In the beginning of 1940 Gödel left Europe permanently to emigrate to USA with his wife. Gödel was not a very political person. He was however suspicious in the eyes of the Nazi officials because he associated with Jewish intellectuals, and thus Gödel lost his appointment as Privatdozent at university. He was even assaulted in the street by a Nazi gang. What was perhaps worse is that he was deemed healthy enough to be eligible for military service, despite his medical issues. All this influenced his decision to emigrate. Obtaining a visa to USA, and getting a permission to leave Germany, was not easy, yet it succeeded with the help of friends such as von Neumann. Due to the war, Gödel and his wife had to make a grueling journey with train through Russia, and then with a boat from Japan across the Pacific Sea to reach San Francisco on 4 March. (Dawson: 139-151.)

2.2 Gödel in Princeton

Gödel's destination in the United States was Princeton, where he had visited many times before. There he took up an appointment as a member of the Institute for Advanced study. He became a permanent member of the institute in 1946 and a professor in 1953. In Princeton Gödel's work
would turn more towards philosophical questions. He would remain there until his death. (Kennedy 2015.)

Notable and well known aspect of Gödel's life in Princeton was his friendship with Albert Einstein. Gödel and Einstein had very different personalities: Gödel was serious and solitary while Einstein gregarious and happy. Yet the two men formed a strong bond. They were often seen walking together discussing philosophy, physics and politics. Both men possessed a rare intellect and it may have been useful and interesting for them to discuss ideas with someone of equal intellect who held different views. It may also be that Einstein recognized that Gödel needed someone to look after him and was willing to do that himself. (Dawson: 176-177.)

Interestingly, although not necessarily because of Einstein’s influence, Gödel did do some revolutionary work concerning Einstein’s own creation: relativity theory. In 1949 Gödel showed that absolute time is not a necessary property of all possible solutions of relativity theory. This means that relativity theory may allow time travel, at least in principle, if certain assumptions about the construction of the universe are true. Later, even close to the end of his life, Gödel is known to have inquired from astronomers if there was empirical evidence that universe was rotating. Rotation might mean that the real universe is similar to the universe that Gödel described. To his disappointment no evidence of the rotation was found at least in Gödel's lifetime. In any case, Gödel considered that the mere possibility of this kind of universe had significant philosophical implications about the nature of time. (Dawson: 177, 181-184.)

Near the end of 1947 Gödel and his wife decided to acquire the citizenship of the United States. Gödel's behavior in his citizenship examination is one of the most famous anecdotes told about him and too good a story, and illustrative of Gödel's character, not to tell here. There are some slightly differing versions of the story. The version presented here is based on Dawson’s book (Dawson: 179-180). Gödel had two of his closest friends to act as witnesses for him in the hearing: Einstein and Oskar Morgenstern, famous for his work with von Neumann concerning game theory. As part of the process Gödel expected to be questioned about the American system of government, like all the applicants for the citizenship are. Gödel's however prepared for the examination much more thoroughly than most people would. Before the day of the examination visibly agitated Gödel told to Morgenstern that he had found an inconsistency in the Constitution that would permit the United States to fall into a dictatorship. Morgenstern and Einstein were afraid that Gödel's citizenship
application might fail if he would say something like that to the examining judge and tried to convince Gödel to stay silent of such matters and just answer the questions briefly. On the day of the examination Einstein and Morgenstern did their best to distract Gödel so he would forget his concerns about the Constitution, for example with Einstein telling jokes. However when the examination begun the judge asked Gödel if he thought that dictatorship like the one in Germany could ever rise in the United States. This was the very question that horrified Einstein and Morgenstern were afraid of, but the very thing that Gödel had hoped to speak about and he started to rant about how he had discovered that dictatorship could indeed rise because of the logical inconsistency of the Constitution. To the relief of Einstein and Morgenstern, the judge was sympathetic and had the good sense to interrupt Gödel's rant before the exchange got too heated. Gödel's citizenship application was approved and he would later remember the judge warmly.

Gödel doesn't seem to have been generally very interested about the developments in computer science. There is however a very interesting letter that he wrote to then terminally ill John von Neumann in 1956. After expressing hope for his friends complete recovery, he presented a mathematical problem for von Neumann's consideration. This problem has later been recognized as the first known statement of what has since come to be called P versus NP problem, a central unsolved question in computer science. Unfortunately von Neumann was then already too ill to respond, and it can only be speculated what he would have had to say about the content of the letter. (Dawson: 205.) According to Sam Buss (1995) Gödel does not seem have realized all the aspects of the theory of NP-completeness. He was however thinking about these issues, in generally correct direction, well before others. Gödel can be interpreted as suggesting in the letter that P =NP might be true, which goes against the modern consensus and would mean that many computing problems have far faster solutions than currently thought. However the opinion of Buss is that Gödel did not intend to say that this was likely, but merely that he couldn't disprove it.

Gödel withdrew more and more from the world during the later years of his life. Still, Gödel did keep working. One thing he consider of particular importance was his work on a logical proof for the existence of God known as 'ontological argument' since the medieval times. Gödel also found troubling Alan Turing's idea that human mind can't transcend mechanical procedures of a Turing machine. Gödel seems have thought that human mind is potentially unlimited, unlike machines, and worked to find a proof for that. (Dawson: 229-232.)
For the whole of his life Gödel suffered from health problems that were generally result of psychiatric disturbance. He was a hypochondriac and occasionally paranoid. Gödel always had an especially difficult relationship with food, and he would grow more and more reluctant to eat during his later years. Although Gödel remained almost unknown outside the mathematical community, late in his life he did receive some wider recognition. Princeton University decided to award him honorary doctorate and in 1975 he received the National Medal of Science. Perhaps because of his health he did not arrive to personally receive neither of these honors, or perhaps he just didn't care anymore. (Dawson: 246-249.)

Gödel retired officially in 1976 and died in Princeton on January 14, 1978 at the age of 71. After a lifetime of struggling with eating and being afraid of food, Gödel finally starved himself to death in his paranoia and died of malnutrition. His wife died three years later. (Kennedy 2015.)

3. Incompleteness and Formal Systems

3.1 Crisis in the Foundations of Mathematics

To understand the significance of Gödel's work, a very short history of modern logic is presented in this paragraph based on Dawson's book (Dawson: 37-52). Gottfried Leibniz, born in 1646, realized that mathematical procedures could be applied to logic. He had an idea of an abstract symbolic algebra and a vision that reasoning might be mechanized, although he made little progress in realizing this vision. Gödel was a great admirer of Leibniz and often read his works. It wasn't, however, before George Boole's work in 1847 that real progress in logic was achieved. In 1879 Gottlob Frege presented his influential formal system. Frege was trying, unsuccessfully, to give mathematics a solid foundation by showing how it can be derived from logic. Around the turn of the century many paradoxes were presented. Perhaps the most famous of these is Russell's Paradox, where Russell showed a flaw in Frege's system. Paradoxes in their part led to a crisis in the foundations of mathematics. In 1900 David Hilbert presented ten problems in mathematics that needed to be solved. The second of those problems was to prove that arithmetic is consistent. Later, in 1920s, Hilbert introduced a more detailed program to clarify the foundations of mathematics and solve the crisis.
It is generally thought that Gödel's incompleteness theorems meant the end of Hilbert's program, although Gödel himself was convinced of that only after Alan Turing precisely defined formal systems (Kennedy 2015). This happened, even though Gödel seems to have provided a negative answer to Hilbert's second problem, he originally set out to advance Hilbert's program and not to destroy it (Dawson: 61). Gödel's work meant that paradoxes would be here to stay.

### 3.2 Incompleteness Theorems

On 7 September 1930, as part of a discussion in a conference in Königsberg, Gödel suggested first time in public that one can give examples of true propositions that are unprovable within arithmetic. Most people present did not realize the importance of what Gödel said. There was however one person who did. Von Neumann drew Gödel aside after the session to discuss Gödel's ideas. After the conference von Neumann worked on these ideas and on 20 November he wrote to Gödel that he had succeeded in establishing the unprovability of consistency. However, by this time Gödel had completed his own paper that included his incompleteness theorems. The full manuscript was received by the publisher only three days before von Neumann wrote his letter. (Dawson: 69-70.)

Although the original formal statements of Gödel's incompleteness theorems are highly technical there is an easily understandable formulation of those theorems in a note added later on 1963 by Gödel to his original paper (Gödel 1931): "...in every consistent formal system that contains a certain amount of finitary number theory there exist undecidable arithmetic propositions and that, moreover, the consistency of any such system cannot be proved in the system." Here one can easily see the basic idea of the theorems. First incompleteness theorem is referenced in the first part of the statement and the second in the latter. Gödel meant this to be a "completely general” version of his incompleteness theorems, and felt that he could give such a version because of Turing's work.

It would be far too difficult and unnecessary to go formally through the complicated elements of Gödel's (1931) proof for his incompleteness theorems. Still, some aspects of his paper need to be noted and are important from the point of view of computer science. Gödel presents in his paper a certain kind of formal version of something like a liar's paradox. Liar's paradox is an ancient antimony that can be stated for example as follows: "This sentence is a lie.” If the sentence is true then it's a lie and if it's a lie then it's true, therefore it's a paradox and it's neither true or false. Gödel
was able to construct a formal arithmetic version of this kind of self-referential paradox by using a technique called Gödel numbering, and thus create an undecidable arithmetic proposition. The exact method used by Gödel is not that important here, but the basic idea is. Gödel showed that any arithmetic statement, and indeed any symbolic presentation, can be interpreted as a number. For a modern computer scientist this may seem like a trivial idea. We are used to the fact that this is how computers store and manipulate information, but in Gödel's time this was not so obvious.

3.3 Influence on Church and Turing

Even though the results of Gödel's 1931 paper were surprising, they were generally approved. There was however at the time some question about how general the results were. The notions of decidability and formal systems needed to be properly defined. (Raatikainen 2015.) Gödel presented a definition of computable functions, his notion of general recursiveness, in a lecture in Princeton in 1934. An important reason for this was to establish the generality of his incompleteness theorems better. The content of this lecture has been later considered by some to be a statement of the Church-Turing thesis. Gödel himself strongly disagreed, for he had not at the time considered it to even be possible to define all possible recursion, even though his general recursion was later shown to be equivalent with formalizations of Church and Turing. (Dawson: 101-102.)

Drawing heavily on Gödel's earlier work, and perhaps on his personal interaction with Gödel in Princeton, Alonzo Church was in 1936 able to give a negative answer to the decision problem, Hilbert's Entscheidungsproblem, and define formal systems technically correctly (Zach 2006). Gödel however found Church's definition unsatisfactory. Gödel considered only the definition of formal systems given by Alan Turing to be adequate and satisfactory (Dawson: 101).

Slightly after Church had published his results, Alan Turing published his own work (1936) where he solved the decision problem by defining so called Turing machines. The influence of Gödel on the work of Turing is perhaps less direct than his influence on Church, after all Gödel and Turing never met. Still the influence is clear. In his paper Turing references Gödel multiple times. However this is mostly in the context of Turing trying to convince reader that his results are different than Gödel's. The clear implication of this is that Turing expects certain similarities of his work and that of Gödel's to be obvious for the reader. Turing himself characterizes the difference to be that Gödel
showed that there are undecidable propositions, while Turing demonstrates that there is no way of knowing which propositions are undecidable. Characterized this way, Turing's paper can be seen as an extension of Gödel's work.

Similarities of Gödel's paper (1931) and Turing's (1936) can be seen especially in the way they use self-reference. While Gödel creates a self-referential arithmetic statement to produce an undecidable statement, Turing creates a self-referential machine to produce the halting problem. Turing also uses here a technique very similar to Gödel numbering. (Bolander 2015.)

4. Conclusion

Kurt Gödel's incompleteness theorems were revolutionary and they changed forever the way we see mathematics. Gödel was guided to his discovery by his philosophical ideas: he was a mathematical Platonist and believed that mathematics and logic represent higher truths that a mathematician can't define but only discover.

Besides his historical significance, Gödel's influence can be seen today directly in some interesting discussions concerning computer science. There are some who argue against the possibility of strong AI based on so called Gödelian argument (Raatikainen 2015). The work of Gregory Chaitin on algorithmic information theory is also something that one should look up if interested in implications of Gödel's theorems. However, because time and space is running out, this report had to concentrate elsewhere.

Gödel was a highly impractical man. His work was theoretical in the extreme and often guided by his unusual philosophical notions about nature of mathematical truth and time. His work didn't have the slightest foreseeable practical application. Yet, it seems that sometimes such work, of such a man, can have consequences that are quite useful: consequences like computer science and everything derived from it.
References


