ON SMALL RAMSEY NUMBERS IN GRAPHS

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Abstract. We give exact values for certain small 2-colour Ramsey numbers in graphs. In particular, we prove that \( R(3, 3) = 6 \) and \( R(4, 4) = 18 \).

In the following, if \( G = (V, E) \) is a graph and \( U \subseteq V \), we denote by \( G[U] \) the subgraph induced by \( U \). For set \( X \), we denote by \([X]^k \) the collection of all subsets of \( X \) with \( k \) elements.

We will now prove some well-known results about certain small Ramsey numbers. The following treatment of the matter along with the theorems and proofs is quite standard; the results are originally from Greenwood and Gleason [1], except for theorem 2, which is quite elementary and had appeared as an exercise in the William Lowell Putnam Mathematical Competition held in March 1953. We use notation from Radziszowski [2].

Definition 1. For \( k, l \geq 2 \), denote by \( R(k, l) \) the smallest number \( N \) such that for all graphs \( G = (V, E) \) with at least \( N \) vertices, \( G \) contains either a \( k \)-clique or an independent set with \( l \) vertices. The values \( R(k, l) \) are called 2-colour Ramsey numbers in graphs.

Now, we want to find out the exact values of \( R(k, l) \) for certain small values of \( k \) and \( l \). We start with an easy case.

Theorem 2. \( R(3, 3) = 6 \).

Proof. First, observe that the 5-cycle \( C_5 \) does not contain a 3-clique or an independent set with 3 vertices. Thus, \( R(3, 3) > 5 \).

Now assume that \( G = (V, E) \) is a graph with \( |V| = 6 \). Let \( u \in V \) be an arbitrary vertex. There are two possible scenarios:

1. The set \( N = \{ v \in V \mid \{u, v\} \in E \} \) has at least three elements. In this case, either the set \( N \) is independent and the theorem holds, or we have two adjacent vertices \( v_1, v_2 \in N \), in which case \( \{u, v_1, v_2\} \) is a clique and the theorem also holds.
2. The set \( \{ v \in V \mid \{u, v\} \in E \} \) has at most two elements. Then by case 1), there is a clique or a independent set of size 3 in the complement graph of \( G \) and thus also in \( G \).

In any case, we have that \( R(3, 3) \leq 6 \). \( \Box \)

To get tight bounds for \( R(4, 4) \), we need to see some more trouble. As a starting point, we observe that \( R(m, 2) = R(2, m) = m \) for all \( m \geq 2 \), because a graph with \( m \) vertices is either \( K_m \) or has two non-adjacent vertices, and on the other hand, \( K_{m-1} \) serves as the proof for the lower bound.

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Lemma 3. For all $k, l \geq 3$, we have that
\[ R(k, l) \leq R(k - 1, l) + R(k, l - 1). \]

Proof. Let $k, l \geq 3$. Let $G = (V, E)$ be a graph with $n = R(k - 1, l) + R(k, l - 1)$ vertices. We show now that there is either a $k$-clique or an independent set with $l$ vertices in $G$.

Fix $u \in V$. Now we define
\[ V_+(u) = \{ v \in V \mid \{u, v\} \in E \} \]
\[ V_-(u) = \{ v \in V \mid \{u, v\} \notin E \}. \]

Observe that $\{u\}, V_+(u)$ and $V_-(u)$ are disjoint and their union is $V$. Thus, (1)
\[ |V_+(u)| + |V_-(u)| = R(k - 1, l) + R(k, l - 1) - 1. \]

This means that we must have $|V_+(u)| \geq R(k - 1, l)$ or $|V_-(u)| \geq R(k, l - 1)$, because otherwise inequality 1 would not hold. Thus, we now have two possible cases:

1. We have $|V_+(u)| \geq R(k - 1, l)$, and therefore $G[V_+(u)]$ either has a $(k - 1)$-clique or an independent set with $l$ vertices. In the latter case, we are done; otherwise, there is $K \subseteq V_+(u)$ such that $K$ is a $(k - 1)$-clique. By the definition of $V_+(u)$, the set $\{u\} \cup K$ is a $k$-clique.

2. We have $|V_-(u)| \geq R(k, l - 1)$. Again, we either have a $k$-clique in $G[V_-(u)]$, in which case the theorem holds, or then there is an independent set $I \subseteq V_-(u)$ with $|I| = l - 1$. In the latter case $\{u\} \cup I$ is an independent set with $l$ vertices.

In some cases, the result of lemma 3 can be improved slightly.

Lemma 4. For all $k, l \geq 3$, if $R(k - 1, l) = 2p$ and $R(k, l - 1) = 2q$, then
\[ R(k, l) \leq R(k - 1, l) + R(k, l - 1) - 1. \]

Proof. Let $k, l \geq 3$ such that $R(k - 1, l) = 2p$ and $R(k, l - 1) = 2q$. Let $G = (V, E)$ be a graph with $n = R(k - 1, l) + R(k, l - 1) - 1 = 2p + 2q - 1$ vertices. Again, we want to show that there is either a $k$-clique or an independent set with $l$ vertices in $G$.

Observe that if there is a vertex $u \in V$ such that $|V_+(u)| \geq R(k - 1, l)$ or $|V_-(u)| \geq R(k, l - 1)$, then we can use same arguments as in the proof of lemma 3 to see that there is a $k$-clique or an independent set with $l$ vertices in $G$. Thus, it is sufficient to show that such $u$ exists.

The problematic case now that it might be that for all $v \in V$, $|V_+(v)| = R(k - 1, l) - 1 = 2p - 1$ and $|V_-(v)| = R(k, l - 1) - 1 = 2q - 1$. Assume that this is in fact the case. In particular, then each vertex has degree $2p - 1$, and thus there are $(2p - 1)(2p + 2q - 1)/2$ edges in $G$. However, $(2p - 1)(2p + 2q - 1)/2$ is not an integer, so this is not possible. 

Lemma 5. $R(4, 4) \leq 18$.

Proof. We have previously seen that
\[ R(4, 2) = 4, \]
\[ R(2, 4) = 4, \]
\[ R(3, 3) = 6. \]
Using lemmas 3 and 4, we get that
\[ R(3, 4) \leq R(2, 4) + R(3, 3) - 1 = 9, \]
\[ R(4, 3) \leq R(3, 3) + R(4, 2) - 1 = 9, \] and
\[ R(4, 4) \leq R(3, 4) + R(4, 3) = 18. \]

\[ \square \]

**Lemma 6.** \( R(4, 4) > 17. \)

**Proof.** We start by defining set
\[ S_{17} = \{ x^2 \mid x \in \mathbb{Z}_{17} \} \setminus \{ 0 \} = \{ 1, 2, 4, 8, 9, 13, 15, 16 \}. \]

Now let \( G = (\mathbb{Z}_{17}, E) \), where
\[ E = \{ (x, y) \in [\mathbb{Z}_{17}]^2 \mid x - y \in S_{17} \}. \]

(See figure [1]) Observe that since in the field \( \mathbb{Z}_{17} \) it holds that \(-1 = 16 = 4^2\), if \( x - y = a^2 \) for some \( a \), then \( y - x = (-1)(x - y) = (4a)^2 \), and thus \( G \) is well-defined.

Now suppose that \( K \subseteq \mathbb{Z}_{17} \) is a 4-clique in \( G \). We may in fact assume that \( 0 \in K \), because otherwise we get such a clique by subtracting the smallest element in \( K \) from all the elements of \( K \). Thus, suppose that \( K = \{ 0, a, b, c \} \); by definition of \( G \), it holds that \( H = \{ a, b, c, a - b, a - c, b - c \} \subseteq S_{17} \). Since \( \mathbb{Z}_{17} \) is a field, \( a^{-1} \) exists. We define \( B = ba^{-1} \) and \( C = ca^{-1} \); these are distinct numbers and different from 1, since \( a, b \) and \( c \) are distinct. Because \( a^{-1} = (n^2)^{-1} \) for some \( n \in \mathbb{Z}_{17} \), by multiplying all elements of \( H \) by \( a^{-1} \), we get that
\[ \{ 1, B, C, 1 - B, 1 - C, B - C \} \subseteq S_{17}. \]

On the other hand, suppose that \( I \subseteq \mathbb{Z}_{17} \) is an independent set in \( G \) with 4 elements. Again, we may assume that \( I = \{ 0, a, b, c \} \). We have now that \( J = \{ a, b, c, a - b, a - c, b - c \} \subseteq \mathbb{Z}_{17} \setminus (S_{17} \cup \{ 0 \}) \). It can be easily verified by testing
all possible cases that if \( x, y \in \mathbb{Z}_{17} \setminus (S_{17} \cup \{0\}) \), then \( xy \in S_{17} \). Thus multiplying all the elements of \( J \) by \( a^{-1} \) we see that 
\[
\{1, B, C, 1 - B, 1 - C, B - C\} \subseteq S_{17},
\]
where again \( B = ba^{-1} \) and \( C = ca^{-1} \).

We have seen that if there is a 4-clique or an independent set with 4 vertices in \( \mathcal{G} \), then there are distinct number \( B, C \in S_{17} \setminus \{1\} \) such that 
\[
\{1, B, C, 1 - B, 1 - C, B - C\} \subseteq S_{17}.
\]
We have that 
\[
\begin{align*}
1 - 2 &= 16 \in S_{17} \\
1 - 4 &= 14 \notin S_{17} \\
1 - 8 &= 10 \notin S_{17} \\
1 - 9 &= 9 \in S_{17} \\
1 - 13 &= 5 \notin S_{17} \\
1 - 15 &= 3 \notin S_{17} \\
1 - 16 &= 1 \in S_{17},
\end{align*}
\]
and thus \( B, C \in \{2, 9, 16\} \). However,
\[
\begin{align*}
B = 9, C = 2 &\implies B - C = 7 \notin S_{17} \\
B = 2, C = 9 &\implies B - C = 10 \notin S_{17} \\
B = 16, C = 2 &\implies B - C = 15 \notin S_{17} \\
B = 2, C = 16 &\implies B - C = 3 \notin S_{17} \\
B = 16, C = 9 &\implies B - C = 7 \notin S_{17} \\
B = 9, C = 16 &\implies B - C = 10 \notin S_{17}.
\end{align*}
\]
It follows that set \( \{1, B, C, 1 - B, 1 - C, B - C\} \) cannot be a subset of \( S_{17} \). Thus, existence of 4-clique or an independent set with 4 vertices would lead to a contradiction and is therefore not possible.

Since \( \mathcal{G} \) is a graph with no 4-clique or independent set of 4 vertices, we have that \( R(4, 4) > 17 \).

Combining the previous lemmas we get the following theorem.

**Theorem 7.** \( R(4, 4) = 18 \). □

**References**