ON SMALL RAMSEY NUMBERS IN GRAPHS

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Abstract. We give exact values for certain small 2-colour Ramsey numbers in graphs. In particular, we prove that $R(3,3) = 6$ and $R(4,4) = 18$.

In the following, if $\mathcal{G} = (V, E)$ is a graph and $U \subseteq V$, we denote by $\mathcal{G}[U]$ the subgraph induced by U. For set X, we denote by $[X]^k$ the collection of all subsets of X with k elements.

We will now prove some well-known results about certain small Ramsey numbers. The following treatment of the matter along with the theorems and proofs is quite standard; the results are originally from Greenwood and Gleason [1], except for theorem 2, which is quite elementary and had appeared as an exercise in the William Lowell Putnam Mathematical Competition held in March 1953. We use notation from Radziszowski [2].

Definition 1. For $k, l \geq 2$, denote by $R(k,l)$ the smallest number N such that for all graphs $\mathcal{G} = (V, E)$ with at least N vertices, $\mathcal{G}$ contains either a k-clique or an independent set with l vertices. The values $R(k,l)$ are called *2-colour Ramsey numbers in graphs.*

Now, we want to find out the exact values of $R(k,l)$ for certain small values of k and l. We start with an easy case.

Theorem 2. $R(3,3) = 6$.

Proof. First, observe that the 5-cycle C_5 does not contain a 3-clique or an independent set with 3 vertices. Thus, $R(3,3) > 5$.

Now assume that $\mathcal{G} = (V, E)$ is a graph with $|V| = 6$. Let $u \in V$ be an arbitrary vertex. There are two possible scenarios:

(1) The set $N = \{v \in V \mid \{u, v\} \in E\}$ has at least three elements. In this case, either the set N is independent and the theorem holds, or we have two adjacent vertices $v_1, v_2 \in N$, in which case $\{u, v_1, v_2\}$ is a clique and the theorem also holds.
(2) The set $\{v \in V \mid \{u, v\} \in E\}$ has at most two elements. Then by case (1), there is a clique or a independent set of size 3 in the complement graph of $\mathcal{G}$ and thus also in $\mathcal{G}$.

In any case, we have that $R(3,3) \leq 6$. □

To get tight bounds for $R(4,4)$, we need to see some more trouble. As a starting point, we observe that $R(m,2) = R(2,m) = m$ for all $m \geq 2$, because a graph with m vertices is either K_m or has two non-adjacent vertices, and on the other hand, K_{m-1} serves as the proof for the lower bound.

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Lemma 3. *For all $k, l \geq 3$, we have that*

$$R(k,l) \leq R(k-1,l) + R(k,l-1).$$

Proof. Let $k, l \geq 3$. Let $\mathcal{G} = (V, E)$ be a graph with $n = R(k-1,l) + R(k,l-1)$ vertices. We show now that there is either a k-clique or an independent set with l vertices in $\mathcal{G}$.

Fix $u \in V$. Now we define

$$V_+(u) = \{v \in V \mid \{u,v\} \in E\}$$
$$V_-(u) = \{v \in V \mid \{u,v\} \notin E\}.$$

Observe that $\{u\}$, $V_+(u)$ and $V_-(u)$ are disjoint and their union is V. Thus,

$$|V_+(u)| + |V_-(u)| = R(k-1,l) + R(k,l-1) - 1. \tag{1}$$

This means that we must have $|V_+(u)| \geq R(k-1,l)$ or $|V_-(u)| \geq R(k,l-1)$, because otherwise inequality 1 would not hold. Thus, we now have two possible cases:

(1) We have $|V_+(u)| \geq R(k-1,l)$, and therefore $\mathcal{G}[V_+(u)]$ either has a $(k-1)$-clique or an independent set with l vertices. In the latter case, we are done; otherwise, there is $K \subseteq V_+(u)$ such that K is a $(k-1)$-clique. By the definition of $V_+(u)$, the set $\{u\} \cup K$ is a k-clique.
(2) We have $|V_-(u)| \geq R(k,l-1)$. Again, we either have a k-clique in $\mathcal{G}[V_-(u)]$, in which case the theorem holds, or then there is an independent set $I \subseteq V_-(u)$ with $|I| = l-1$. In the latter case $\{u\} \cup I$ is an independent set with l vertices.

□

In some cases, the result of lemma 3 can be improved slightly.

Lemma 4. *For all $k, l \geq 3$, if $R(k-1,l) = 2p$ and $R(k,l-1) = 2q$, then*

$$R(k,l) \leq R(k-1,l) + R(k,l-1) - 1.$$

Proof. Let $k, l \geq 3$ such that $R(k-1,l) = 2p$ and $R(k,l-1) = 2q$. Let $\mathcal{G} = (V,E)$ be a graph with $n = R(k-1,l) + R(k,l-1) - 1 = 2p + 2q - 1$ vertices. Again, we want to show that there is either a k-clique or an independent set with l vertices in $\mathcal{G}$.

Observe that if there is a vertex $u \in V$ such that $|V_+(u)| \geq R(k-1,l)$ or $|V_-(u)| \geq R(k-1)$, then we can use same arguments as in the proof of lemma 3 to see that there is a k-clique or an independent set with l vertices in $\mathcal{G}$. Thus, it is sufficient to show that such u exists.

The problematic case now that it might be that for all $v \in V$, $|V_+(v)| = R(k-1,l) - 1 = 2p - 1$ and $|V_-(v)| = R(k,l-1) - 1 = 2q - 1$. Assume that this is in fact the case. In particular, then each vertex has degree $2p-1$, and thus there are $(2p-1)(2p+2q-1)/2$ edges in $\mathcal{G}$. However, $(2p-1)(2p+2q-1)/2$ is not an integer, so this is not possible. □

Lemma 5. *$R(4,4) \leq 18$.*

Proof. We have previously seen that

$$R(4,2) = 4,$$
$$R(2,4) = 4, \text{ and}$$
$$R(3,3) = 6.$$

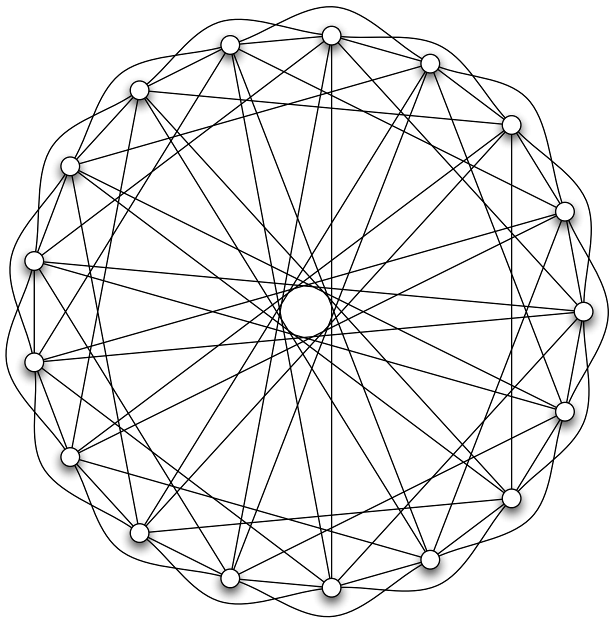

FIGURE 1. Graph $\mathcal{G}$.

Using lemmas 3 and 4, we get that

$$R(3,4) \le R(2,4) + R(3,3) - 1 = 9,$$
$$R(4,3) \le R(3,3) + R(4,2) - 1 = 9, \text{ and}$$
$$R(4,4) \le R(3,4) + R(4,3) = 18.$$

□

Lemma 6. $R(4,4) > 17$.

Proof. We start by defining set

$$S_{17} = \{x^2 \mid x \in \mathbb{Z}_{17}\} \setminus \{0\} = \{1,2,4,8,9,13,15,16\}.$$

Now let $\mathcal{G} = (\mathbb{Z}_{17}, E)$, where

$$E = \{\{x,y\} \in [\mathbb{Z}_{17}]^2 \mid x - y \in S_{17}\}.$$

(See figure 1.) Observe that since in the field $\mathbb{Z}_{17}$ it holds that $-1 = 16 = 4^2$, if $x - y = a^2$ for some a, then $y - x = (-1)(x-y) = (4a)^2$, and thus $\mathcal{G}$ is well-defined.

Now suppose that $K \subseteq \mathbb{Z}_{17}$ is a 4-clique in $\mathcal{G}$. We may in fact assume that $0 \in K$, because otherwise we get such a clique by subtracting the smallest element in K from all the elements of K. Thus, suppose that $K = \{0,a,b,c\}$; by definition of $\mathcal{G}$, it holds that $H = \{a,b,c,a-b,a-c,b-c\} \subseteq S_{17}$. Since $\mathbb{Z}_{17}$ is a field, a^{-1} exists. We define $B = ba^{-1}$ and $C = ca^{-1}$; these are distinct numbers and different from 1, since a, b and c are distinct. Because $a^{-1} = (n^2)^{-1}$ for some $n \in \mathbb{Z}_{17}$, by multiplying all elements of H by a^{-1}, we get that

$$\{1, B, C, 1-B, 1-C, B-C\} \subseteq S_{17}.$$

On the other hand, suppose that $I \subseteq \mathbb{Z}_{17}$ is an independent set in $\mathcal{G}$ with 4 elements. Again, we may assume that $I = \{0,a,b,c\}$. We have now that $J = \{a,b,c,a-b,a-c,b-c\} \subseteq Z_{17} \setminus (S_{17} \cup \{0\})$. It can be easily verified by testing all possible cases that if $x, y \in Z_{17} \setminus (S_{17} \cup \{0\})$, then $xy \in S_{17}$. Thus multiplying all the elements of J by a^{-1} we see that

$$\{1, B, C, 1-B, 1-C, B-C\} \subseteq S_{17},$$

where again $B = ba^{-1}$ and $C = ca^{-1}$.

We have seen that if there is a 4-clique or an independent set with 4 vertices in $\mathcal{G}$, then there are distinct number $B, C \in S_{17} \setminus \{1\}$ such that

$$\{1, B, C, 1-B, 1-C, B-C\} \subseteq S_{17}.$$

We have that

$$\begin{aligned} 1-2 &= 16 \in S_{17} \\ 1-4 &= 14 \notin S_{17} \\ 1-8 &= 10 \notin S_{17} \\ 1-9 &= 9 \in S_{17} \\ 1-13 &= 5 \notin S_{17} \\ 1-15 &= 3 \notin S_{17} \\ 1-16 &= 1 \in S_{17}, \end{aligned}$$

and thus $B, C \in \{2, 9, 16\}$. However,

$$\begin{aligned} B = 9, C = 2 &\Rightarrow B - C = 7 \notin S_{17} \\ B = 2, C = 9 &\Rightarrow B - C = 10 \notin S_{17} \\ B = 16, C = 2 &\Rightarrow B - C = 15 \notin S_{17} \\ B = 2, C = 16 &\Rightarrow B - C = 3 \notin S_{17} \\ B = 16, C = 9 &\Rightarrow B - C = 7 \notin S_{17} \\ B = 9, C = 16 &\Rightarrow B - C = 10 \notin S_{17}. \end{aligned}$$

It follows that set $\{1, B, C, 1-B, 1-C, B-C\}$ cannot be a subset of S_{17}. Thus, existence of 4-clique or an independent set with 4 vertices would lead to a contradiction and is therefore not possible.

Since $\mathcal{G}$ is a graph with no 4-clique or independent set of 4 vertices, we have that $R(4, 4) > 17$. □

Combining the previous lemmas we get the following theorem.

Theorem 7. $R(4, 4) = 18$. □

References

1. R.E. Greenwood and A.M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math **7** (1955), no. 1.
2. S.P. Radziszowski, *Small Ramsey numbers*, Electronic Journal of Combinatorics **1** (1994), last updated 2009.