BOUNDING THE NUMBER OF CONJUGACY CLASSES OF MAXIMAL SUBGROUPS IN CLASSICAL GROUPS

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The aim

- Let $G$ be a group, $K$ a field, $V$ an $n$-dimensional vector field over $K$.
- The aim is to study the maximal subgroups of classical groups.
- Aschbacher’s Theorem divides the maximal subgroups of finite classical groups into “geometrical” ja “irregular” types.
- We use representation theory to study the irregular maximal subgroups.
- More precisely: we try to bound the number of conjugacy classes of irregular maximal subgroups.
Representations (1)

- The *general linear group* $\text{GL}(V)$ consists of invertible linear transformations of $V$.
- A *representation* of $G$ is a homomorphism $G \to \text{GL}(V)$, i.e., a linear action of $G$ in the space $V$.
- The image of a representation is a subgroup of $\text{GL}(V)$.
- Useful fact 1: Having representations makes it possible to use linear algebra in the study of $G$ (matrices, eigenvalues, determinants etc.).
- Useful fact 2: By listing representations one finds subgroups of $\text{GL}(V)$. 
Representations (2)

- The *dimension* of a representation is the dimension of $V$.
- Two representations are *equivalent* if they are related by a linear transformation of $V$.
- That is, $\varphi \sim \psi$ if there is some $T \in GL(V)$ such that
  $$\psi(g)(v) = T^{-1}[\varphi(g)(Tv)]$$
  for all $v \in V$ and $g \in G$.
- A representation is *irreducible* if its image does not stabilise a subspace of $V$.
- If $|G|$ is finite, there is a finite number of inequivalent irreducible representations.
Classical groups (1)

- Subgroups of $GL(V)$ that preserve a bilinear form $B : V \times V \to K$.
- Trivial form $B(x, y) = 0 \sim GL(V)$ (linear group)
- Symmetric form $B(x, y) = B(y, x) \sim O(V)$ (orthogonal group)
- Alternating form $B(x, y) = -B(y, x) \sim Sp(V)$ (symplectic group)
- Hermitian form $B(x, y) = \overline{B(y, x)} \sim U(V)$ (unitary group, the bar indicates an automorphism of the scalar field)
- The choice of the form is important: for example there are many non-isomorphic orthogonal groups.
Classical groups (2)

- Subgroups with determinant 1 are called *special*.
- For example the special orthogonal group $SO(V)$ consists of rotations of $V$.
- Quotienting over the scalar elements produces *projective* groups.
- E.g. the projective special linear group $PSL(V)$ consists of linear transformations of $V$ having determinant 1, where $g$ and $\lambda g$ are identified for all $\lambda \in K$.
- Projective special groups are usually simple, that is, without non-trivial quotients.
Aschbacher’s Theorem (1)

- Deals with finite classical groups, i.e., $V$ is finite (as well as $K$).
- Assume $G_0$ is a finite simple classical group and $G_0 \trianglelefteq G < \text{Aut}(G_0)$.
- Let $M$ be a maximal subgroup of $G$, such that $G_0 \ntriangleleft M$.
- Either a) $M$ belongs to one of eight so-called “geometrical” classes, typically consisting of stabilisers of a substructure of $V$,
- or b) $M$ normalises a simple subgroup $M_0 < G$ and $M_0$ acts irreducibly in $V$ ($M$ is then “irregular”).
- Aschbacher’s classes form a classification of the geometrical subgroups; groups in different classes are not conjugate to each other; the conjugacy within each class is described in Kleidman & Liebeck: The subgroup structure of the finite classical groups.
Aschbacher’s Theorem (2)

- What if $M$ is not “geometrical”? 
- The conjugates of $M$ are determined by the conjugates of $M_0$.
- As $M_0$ is a simple subgroup of $PGL(V)$, it is an image of a representation of a finite quasisimple group.
- If two representations are equivalent, the corresponding subgroups are conjugates.
- Conclusion: The number of conjugacy classes of irregular maximal subgroups can be bounded by bounding the number of $\dim(V)$-dimensional representations of finite (quasi)simple groups.
Finite quasisimple groups

- Finite simple groups are classified:
  - (1) Cyclic groups of prime order (abelian).
  - (2) Alternating groups $\text{Alt}(d)$, when $d \geq 5$.
  - (3) Groups of Lie type (classical and exceptional).
  - (4) Sporadic groups (26 groups).
- Each type has its own representation theory.
- Quasisimple groups are central extensions of simple groups; for each simple group there is a finite number of such extensions.
Representation growth

- Let $H$ be a finite (quasi)simple group.
- Let $r_n(H)$ denote the number of $n$-dimensional irreducible representations of $H$.
- Divide the finite simple groups into finitely many classes $\mathcal{H}_i$ (e.g. alternating, linear, orthogonal...).
- For each $i$, try to bound the sum

$$s_n(\mathcal{H}_i) = \sum_{H \in \mathcal{H}_i} r_n(H).$$
Examples of bounds of representation growth

<table>
<thead>
<tr>
<th>groups</th>
<th>bound</th>
<th>reference</th>
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<tbody>
<tr>
<td>$H \in \text{Alt}$</td>
<td>$r_n(H) &lt; n^{2.5}$</td>
<td>[1]</td>
</tr>
<tr>
<td>$H \in \text{SL}$ (same characteristic)</td>
<td>$r_n(H) &lt; n^{3.8}$</td>
<td>[1]</td>
</tr>
<tr>
<td>$\mathcal{H} = \text{SL}$ (different characteristic)</td>
<td>$s_n(\mathcal{H}) &lt; 2,72n$</td>
<td>[2]</td>
</tr>
<tr>
<td>$\mathcal{H} = \text{U}$ (different characteristic)</td>
<td>$s_n(\mathcal{H}) &lt; 2,89n$</td>
<td>[2]</td>
</tr>
<tr>
<td>$\mathcal{H} = \text{Lie}$ (different characteristic)</td>
<td>$s_n(\mathcal{H}) &lt; 15n$</td>
<td>[2]</td>
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</tbody>
</table>

The result

- The number of conjugacy classes of maximal subgroups of finite classical groups (geometrical and irregular) is at most
  \[ 2n^{5,2} + n \log_2 \log_2 q, \]
  where \( n \) is the dim of \( V \) and \( q \) the size of the scalar field \( K \). [H]
- The same method can be used in complex vector spaces, by studying representations over the complex numbers.
- With classical groups over the complex numbers, Aschbacher’s Theorem is replaced by the Liebeck–Seitz Theorem.
THANK YOU