BOUNDING THE NUMBER OF CONJUGACY CLASSES OF MAXIMAL SUBGROUPS IN CLASSICAL GROUPS

Jokke Häsä
Imperial College London

Postgraduate Group Theory Conference 2012
The University of York

The aim

- Let G be a group, K a field, V an n-dimensional vector field over K.
- The aim is to study the maximal subgroups of classical groups.
- Aschbacher's Theorem divides the maximal subgroups of finite classical groups into "geometrical" ja "irregular" types.
- We use representation theory to study the irregular maximal subgroups.
- More precisely: we try to bound the number of conjugacy classes of irregular maximal subgroups.

Representations (1)

- The *general linear group* $GL(V)$ consists of invertible linear transformations of V.
- A *representation* of G is a homomorphism $G \to GL(V)$, i.e., a linear action of G in the space V.
- The image of a representation is a subgroup of $GL(V)$.
- Useful fact 1: Having representations makes it possible to use linear algebra in the study of G (matrices, eigenvalues, determinants etc.).
- Useful fact 2: By listing representations one finds subgroups of $GL(V)$.

Representations (2)

- The *dimension* of a representation is the dimension of V.
- Two representations are *equivalent* if they are related by a linear transformation of V.
- That is, $\varphi \sim \psi$ if there is some $T \in GL(V)$ such that

$$\psi(g)(v) = T^{-1}[\varphi(g)(Tv)]$$

  for all $v \in V$ and $g \in G$.
- A representation is *irreducible* if its image does not stabilise a subspace of V.
- If $|G|$ is finite, there is a finite number of inequivalent irreducible representations.

Classical groups (1)

- Subgroups of $GL(V)$ that preserve a *bilinear form* $B\colon V \times V \to K$.
- Trivial form $B(x,y) = 0 \rightsquigarrow GL(V)$ (linear group)
- Symmetric form $B(x,y) = B(y,x) \rightsquigarrow O(V)$ (orthogonal group)
- Alternating form $B(x,y) = -B(y,x) \rightsquigarrow Sp(V)$ (symplectic group)
- Hermitian form $B(x,y) = \overline{B(y,x)} \rightsquigarrow U(V)$ (unitary group, the bar indicates an automorphism of the scalar field)
- The choice of the form is important: for example there are many non-isomorphic orthogonal groups.

Classical groups (2)

- Subgroups with determinant 1 are called *special*.
- For example the special orthogonal group $SO(V)$ consists of rotations of V.
- Quotienting over the scalar elements produces *projective* groups.
- E.g. the projective special linear group $PSL(V)$ consists of linear transformations of V having determinant 1, where g and λg are identified for all $\lambda \in K$.
- Projective special groups are usually simple, that is, without non-trivial quotients.

Aschbacher's Theorem (1)

- Deals with finite classical groups, i.e., V is finite (as well as K).
- Assume G_0 is a finite simple classical group and $G_0 \trianglelefteq G < \mathrm{Aut}(G_0)$.
- Let M be a maximal subgroup of G, such that $G_0 \not\leq M$.
- Either **a)** M belongs to one of eight so-called "geometrical" classes, typically consisting of stabilisers of a substructure of V,
- or **b)** M normalises a simple subgroup $M_0 < G$ and M_0 acts irreducibly in V (M is then "irregular").
- Aschbacher's classes form a classification of the geometrical subgroups; groups in different classes are not conjugate to each other; the conjugacy within each class is described in *Kleidman & Liebeck: The subgroup structure of the finite classical groups*.

Aschbacher's Theorem (2)

- What if M is not "geometrical"?
- The conjugates of M are determined by the conjugates of M_0.
- As M_0 is a simple subgroup of $PGL(V)$, it is an image of a representation of a finite quasisimple group.
- If two representations are equivalent, the corresponding subgroups are conjugates.
- Conclusion: The number of conjugacy classes of irregular maximal subgroups can be bounded by bounding the number of dim(V)-dimensional representations of finite (quasi)simple groups.

Finite quasisimple groups

- Finite simple groups are classified:
- (1) Cyclic groups of prime order (abelian).
- (2) Alternating groups Alt(d), when $d \geq 5$.
- (3) Groups of Lie type (classical and exceptional).
- (4) Sporadic groups (26 groups).
- Each type has its own representation theory.
- Quasisimple groups are central extensions of simple groups; for each simple group there is a finite number of such extensions.

Representation growth

- Let H be a finite (quasi)simple group.
- Let $r_n(H)$ denote the number of n-dimensional irreducible representations of H.
- Divide the finite simple groups into finitely many classes $\mathcal{H}_i$ (e.g. alternating, linear, orthogonal...).
- For each i, try to bound the sum

$$s_n(\mathcal{H}_i) = \sum_{H \in \mathcal{H}_i} r_n(H).$$

Examples of bounds of representation growth

groups	bound	reference
$H \in \mathsf{Alt}$	$r_n(H) < n^{2,5}$	[1]
$H \in SL$ (same characteristic)	$r_n(H) < n^{3,8}$	[1]
$\mathcal{H} = SL$ (different characteristic)	$s_n(\mathcal{H}) < 2{,}72n$	[2]
$\mathcal{H} = U$ (different characteristic)	$s_n(\mathcal{H}) < 2{,}89n$	[2]
$\mathcal{H} = \mathsf{Lie}$ (different characteristic)	$s_n(\mathcal{H}) < 15n$	[2]

[1] Guralnick, Larsen, Tiep: Representation growth in positive characteristic and conjugacy classes of maximal subgroups, 2010
[2] H: Growth of cross-characteristic representations of finite quasisimple groups of Lie type, 2011 (submitted)

The result

- The number of conjugacy classes of maximal subgroups of finite classical groups (geometrical and irregular) is at most

$$2n^{5,2} + n \log_2 \log_2 q,$$

where n is the dim of V and q the size of the scalar field K. [H]
- The same method can be used in complex vector spaces, by studying representations over the complex numbers.
- With classical groups over the complex numbers, Aschbacher's Theorem is replaced by the Liebeck–Seitz Theorem.

THANK YOU