RUBIK'S CUBE FROM A GROUP-THEORETICAL VIEWPOINT

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1. Introduction

Ernő Rubik's "Magic Cube" has, since its invention in 1974 and licensing in 1980, become an internationally famous bestseller toy. Nowadays almost everyone seems to know what the cube looks like and how it feels in their hands. Many will even have tried to solve at least one face. More enthusiastic cubists have themselves come up with methods to solve larger and larger parts of the puzzle, and there are numerous well-known algorithms available for different purposes: from easy-to-learn methods to quick solving.

It is possible to solve the cube very quickly, using sophisticated algorithms which rely on remembering lots of different move sequences corresponding to different positions of the puzzle's pieces. These positions can, in turn, be determined by a quick glance at the cube. The World Cube Association keeps record of the official quickest times for different types of cubes and for different rulesets. The record holder for quickest time of solving the normal $3\times3\times3$ cube is Feliks Zemdegs with 6.77 seconds. For my personal interest, I might mention that the current record for solving both $4\times4\times4$ and $5\times5\times5$ cubes blindfolded, belongs to Ville Seppänen, with 4 minutes 42.34 seconds and 10 minutes 25 seconds, respectively. Also the record for feet-solving a normal cube belongs to a Finn: Anssi Vanhala solved the cube using only his feet in 36.72 seconds in 2009.

Despite the puzzle's popularity, few people are aware of the mathematical structure of the Rubik's cube. Even mathematicians themselves seem to be unaware of its mysteries, although the basic structure is very easily understood in terms of group theory, and there are many books dealing with the subject. In this context, one has to mention the puzzle-master David Singmaster from London South Bank University, who has published *Notes on Rubik's magic cube* (Enslow Pub Inc, 1981) and *Handbook of Cubik Math* (co-author Alexander Frey, The Lutterworth Press, 1987).

Studying the group structure of the cube may not lead to the most profound advancements of modern mathematics, but when it comes to pedagogical devices, there are at least two great benefits in using the puzzle to demonstrate the elementary notions of group theory. Firstly, seeing such a difficult problem as solving the Rubik's cube broken into more or less trivial pieces by the use of algebraic methods motivates the study of these methods and makes one want to try to applying them to other similar problems. Secondly, the somewhat abstract concepts like normal subgroups and conjugation become literally tangible when they are interpreted as certain moves on the puzzle. Once the student has clear concrete analogies for the basic concepts, it is then much easier to proceed to more complicated structures.

In 2008, I designed and gave a course called "Rubik's cube from a group-theoretical viewpoint" in the Department of Mathematics and Statistics of the University of Helsinki. The course received a lot of positive feedback from the students, and at least two from those attending the course were later encouraged to write Bachelor's theses directly related to the presented material. The course was given again in the autumn term of 2010, and the Finnish lecture material is publicly available in the internet.

This presentation is based on the lecture material and ideas formed during the course and afterwards.

2. Structure of the puzzle

Each face of the cube consists of nine cubic *pieces*, including 4 *corner pieces*, 4 *side pieces* and 1 *centre piece*. The faces can be rotated about their centre, and this is the only legal operation to be performed on the puzzle. The *middle layers* that lie between two side faces can also be rotated about their centre, but this move will be considered as a combination of rotations of the parallel side faces, adding a rotation of the whole cube. Naturally, all movements – rotations and translations – of the whole cube in the ambient space will be disregarded, as they have no effect on solving the puzzle.

Clockwise ninety-degree rotations of the side faces are called *basic moves*. Any combination of the basic moves is called a *move* or a *legal move*. Performing a move on the cube leaves the cube in a certain *state* that describes the arrangement of the cube's parts. Two moves are considered the same if they leave the cube in the same final state. The moves are notated by a system coined by David Singmaster, using the letters U, D, F, B, L and R to describe the basic moves of each side (see figure 1). The rotations of the middle layers are denoted U_S, F_S and L_S, so that $U_S = UD^{-1}$, $F_S = FB^{-1}$ and $L_S = LR^{-1}$ after discarding a rotation of the whole cube.

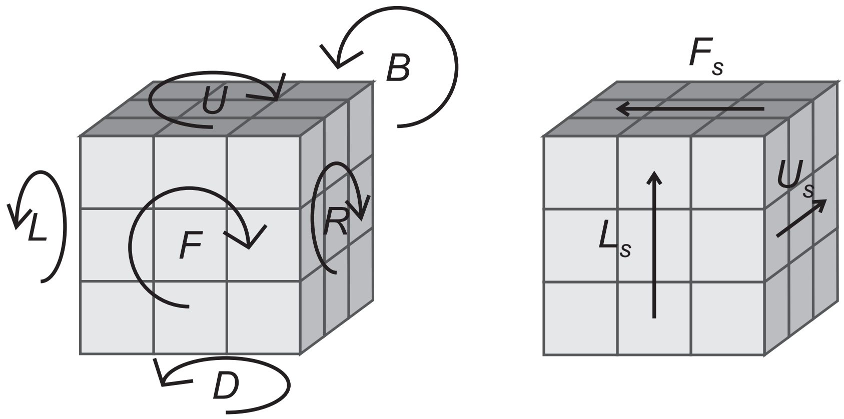

Figure 1. The basic moves and the so-called 'slice' moves of the middle layers

In the *initial state* each side of the cube has a unique solid colour. The colourings vary a little depending on the manufacturer. The sides of each piece are coloured by *stickers*: the corner pieces have 3 stickers each, the side pieces have 2 and the centre piece has only 1. It is important to notice that basic moves, as they only rotate the side faces about their centre pieces, do not change the positions of the centre stickers. As also the moves of the middle layers can be considered as combinations of moves on the side faces, we can conclude that a legal move never changes the positions of the centre stickers, so they can be used as determining which side had which colour in the initial state.

The basic structure underlying the puzzle is the permutation group of all the stickers (not including the centre stickers, because they are not considered to be moving about). This permutation group can be identified with S_{48}. We define the *Rubik's group*, denoted $\mathbb{R}$, to be the subgroup consisting of all permutations induced by legal moves – or alternatively, the subgroup generated by the basic moves. The aim can now be formulated in group-theoretic terms:

> Find a way to express any element of $\mathbb{R}$ as a product of the given generators, i.e. the basic moves.

Because every basic move is easily inverted, we can then bring any state back into the initial state by applying these inverted moves.

In order to be able to talk about the structure of $\mathbb{R}$, the students first need to review the notation used with permutations, in particular the *cycle notation*. It is also important to introduce the concept of *sign*. These are the basic tools that will be used throughout the course.

3. A quotient group

First thing that can be done to ease the solving process is to find a nice quotient group that can be attacked first.

At this point, it may be necessary to remind the students of the definitions of normal subgroup and quotient group. It is, however, equally important to explain why these concepts are important. In this context, I usually try to emphasize the idea that in a quotient structure, the details of the original structure are somewhat blurred. Omitting details, it is often easier to approach a problem, and if some progress is made this way, we may then look at the details again.

The normal subgroup that is chosen for the puzzle consists of those moves that do not move the pieces from their positions, but may only change their orientations. This subgroup is denoted $\mathbb{R}_o$, and called the *orientational group*. It is easy to see that $\mathbb{R}_o$ is really normal, because if σ^{-1} moves a piece to another position, and τ does not change positions, applying σ again (from the left) to the product $\tau\sigma^{-1}$ brings each piece back to its original position (see figure 2).

The elements in a given coset of $\mathbb{R}_o$ all move the pieces in the same way, so the quotient group $\mathbb{R}_p = \mathbb{R}/\mathbb{R}_o$, called the *positional group*, describes the way the pieces are moved about. In effect, we cease to care about the orientations of the individual pieces, but are just interested in bringing the pieces to their correct positions.

The solution has now been divided in two (see figure 3): First bring any state $[\sigma]$ in the positional group to the (coset of) the initial state $[\mathrm{id}] = \mathbb{R}_o$. Then look at the resulting element in the orientational group, and bring that back to the identity. The sequence of basic moves needed is the solution.

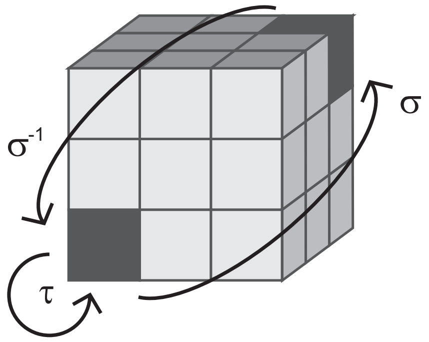

FIGURE 2. Proving that the orientational group is normal

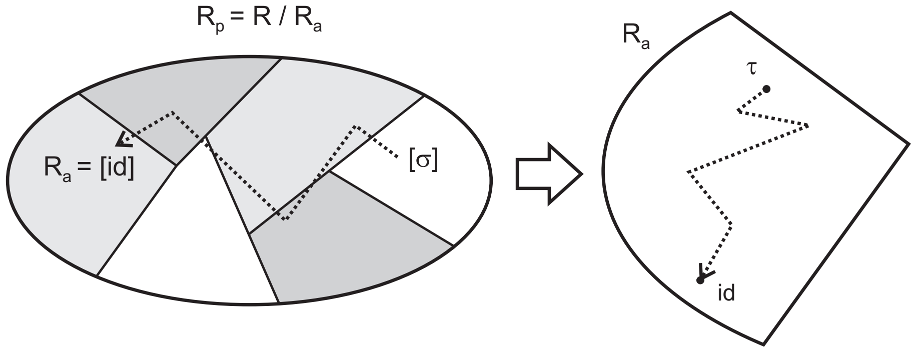

FIGURE 3. The solution is divided into two steps. (Here the orientational group is denoted R_a.)

4. Three-cycles in the positional group

At this point we need to introduce some algorithms so that we can get our hands on things. How the algorithms are obtained is explained later in the course. The first move sequence results in a three-cycle of three corner pieces lying on a same side.

After learning the three-cycle on given corner pieces, the natural question is: "Can we form any desired three-cycle, and how?" This leads to the concept of conjugation. When a group acts on an object, conjugating an element sends the action of that element to another part of the object. This is particularly easy to see in a permutation group, as the conjugate of a cycle is another cycle of the same form, only with the elements in the cycle changed according to the conjugating element. In figure 4, conjugation is used to convert the known three-cycle on the cube into another one.

As a side note, I think it is quite remarkable that the first sophisticated thing people usually learn to do on the cube, is precisely conjugation. Typically they need to move a piece into some position without disturbing another piece already placed, and so they first move that piece out of the way, and afterwards bring it back to its original position.

By going through all possible combinations of three corner pieces, it can be shown that every three-cycle is possible on the corner pieces. This then leads to the fact that we can perform *any even permutation* on the corner pieces.

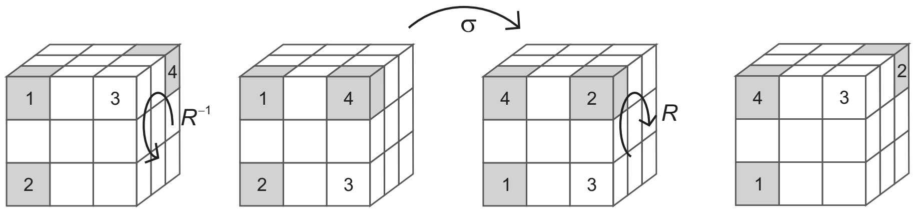

FIGURE 4. Conjugating $\sigma = (123)$ by R to obtain another three-cycle (124)

Next, we show the same thing on the edge pieces: we give a move sequence for a particular three-cycle, and then, going through all combinations of three side pieces, we show that all three-cycles are legal.

5. Product groups

It is evident that moves that only affect corner stickers are completely independent of those that only affect edge stickers. In mathematical terms, we can say that these kinds of moves commute with each other. It then follows that the Rubik's group contains a direct product of $\mathbb{R}_c$, the legal moves that only permute the corner stickers (like the three-cycle that was taught earlier), and $\mathbb{R}_e$, the legal moves that only permute the edge stickers. These are called the *corner* and *edge* subgroups, respectively.

The students are usually familiar with cartesian products of groups, but will not have heard of "inner" direct products. They will be given the definitions and shown the equivalence between inner and outer direct products. It will also be proved that the groups $\mathbb{R}_c$ and $\mathbb{R}_e$ form a direct product inside $\mathbb{R}$. The importance of this product group in terms of the puzzle is that any move inside the product can be broken into two legal moves: one operating on the corner stickers and the other on the edge stickers. Thus, the product group gives a way of determining, for example, the limits of handling corner and edge stickers independently: if a move is not in the product, it can not be produced in this way.

To be able to extract some useful information from the product group $\mathbb{R}_c \times \mathbb{R}_e$, we need to show that the product structure carries on to the quotient structure we have been using. We first define the *orientational subgroups* of $\mathbb{R}_c$ and $\mathbb{R}_e$:

$$\mathbb{R}_{co} = \mathbb{R}_c \cap \mathbb{R}_o \quad \text{and} \quad \mathbb{R}_{eo} = \mathbb{R}_e \cap \mathbb{R}_o.$$

Then we need the following simple theorem.

Theorem 1. *Assume $H, K \leq G$, and $N \trianglelefteq G$. Then $H \cap N \trianglelefteq H$ and $K \cap N \trianglelefteq K$.*

By the above theorem, we can form the *positional groups* of the corner and edge groups:

$$\mathbb{R}_{cp} = \mathbb{R}_c/\mathbb{R}_{co} \quad \text{and} \quad \mathbb{R}_{ep} = \mathbb{R}_e/\mathbb{R}_{eo}.$$

By definition, the positional corner group consists of moves on the corner stickers, disregarding the orientation of each piece. Next, we show that instead of this definition, it is equivalent to think about the permutations in the positional group $\mathbb{R}_p$ that only affect the corner pieces.

Theorem 2. *a) The quotient groups $\mathbb{R}_{cp}$ and $\mathbb{R}_{ep}$ are isomorphic to some subgroups of the positional group $\mathbb{R}_p$. Moreover, the isomorphism are such that $\varphi_1 : [\sigma]_c \mapsto [\sigma]$ for all $\sigma \in \mathbb{R}_c$, and $\varphi_2 : [\tau]_e \mapsto [\tau]$ for all $\tau \in \mathbb{R}_e$.*

b) The images of $\mathbb{R}_{cp}$ and $\mathbb{R}_{ep}$ under the isomorphism given in part a) form a direct product in the group $\mathbb{R}_p$.

We will henceforth identify the product group $\mathbb{R}_{cp} \times \mathbb{R}_{ep}$ with its image inside $\mathbb{R}_p$. With some useful information regarding this product group, we are able to completely solve the positional group. For the next theorem, we use the notation described in the following: Since every permutation of the pieces (not just any legal one) can be written uniquely as a commuting product of a permutation γ of the corner pieces and a permutation ε of the edge pieces, we may write them as a pair $\sigma = (\gamma, \varepsilon)$.

Theorem 3. *Assume that $\sigma = (\gamma, \varepsilon) \in \mathbb{R}_p$. If $\operatorname{sign}(\gamma) = 1$ or $\operatorname{sign}(\varepsilon) = 1$, then both γ and ε belong to the positional group $\mathbb{R}_p$. Moreover, the index of $\mathbb{R}_{cp} \times \mathbb{R}_{ep}$ inside the positional group is $[\mathbb{R}_p : \mathbb{R}_{cp} \times \mathbb{R}_{ep}] = 2$.*

The proof depends on the fact that each basic move is a four-cycle on both corner and edge pieces. Hence the signs of corner and edge permutations must always be equal for each legal move. On the other hand, we know that we can perform any even permutation on the corner pieces as well as on the edge pieces.

Now the positional group can be solved as follows: Check whether the given state is in the positional product group or not. If not, do any basic move to bring it there. Then use the learned three-cycles and their conjugates to solve the corners and edges in any desired order.

6. How to produce nice move sequences

Finally, we come back to the problem of finding the move sequences that were presented earlier.

Algebraically put, the main difficulty in solving the cube is that the basic moves are highly uncommutative. Doing a few basic moves in row tends to scramble the cube irrecoverably, unless one remembers the exact order in which they were performed.

One way of producing "small" moves that will not affect most of the stickers is to find two moves that very nearly commute, and take their commutator. Of course, this can only produce even permutations, but it just so happens that in the positional group we are able to obtain the smallest – and hence all – even permutations. The key result is the following theorem, where the *support* of a permutation σ is defined to be the set of elements that are not fixed under σ.

Theorem 4. *Let σ and τ be permutations of a set X. If the intersection of their supports is a singleton, that is, $\operatorname{supp}(\sigma) \cap \operatorname{supp}(\tau) = \{x\}$ for some $x \in X$, then*

$$[\sigma, \tau] = \big(x \ \sigma(x) \ \tau(x)\big).$$

In the positional group, to produce a three-cycle $(x\ y\ z)$, one has to arrange things so that a move σ will take the piece from x to y, a move τ will take x to z, and the only piece that is moved by both σ and τ, is x. For example, we might take $\sigma = U$ and $\tau = RDR^{-1}$. The supports of these permutations are described in figure 5.

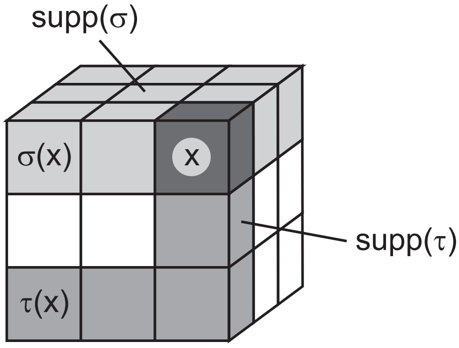

FIGURE 5. The supports of $\sigma = U$ and $\tau = RDR^{-1}$. Some of the support of τ is hidden behind the cube.

In the orientational group, things are not so easy, as it is not possible for a move to affect only one sticker in a piece and fix the others. It is thus impossible to produce three-cycles in the way described above. The same principle of commutators is, however, still valid. We just need to find two moves whose common support is *as small as possible*, and take their commutator. It turns out that one of them has to move pieces about, as the orientational group is commutative. On the other hand, if σ is any move, and τ is in $\mathbb{R}_o$, then the commutator $\sigma\tau\sigma^{-1}\tau^{-1}$ is in $\mathbb{R}_o$ because $\mathbb{R}_o$ is normal.

For example, to rotate a corner piece x on the top side, one might do the following: Take $\sigma = U$. Then build up a sequence of moves τ that will not touch any other piece on the top side except x, and will rotate x in place. (This can be done by conjugating x to the bottom side where it can be rotated and then brought back.) The resulting commutator of σ and τ will actually rotate two corner pieces in opposite directions, but this is the best we can do.

7. Solving the orientational group

Now that the positional group is solved, we may concentrate on the orientational group. This is an abelian group isomorphic to a subgroup of $C_3^8 \times C_2^{12}$. The commutator approach will give us a move sequence that rotates two adjacent corner pieces in opposite directions, and another sequence that "flips" two neighbouring edge pieces. These will enable us to work through the corners (or edges) one by one, so that in the end only one corner (or edge) may possibly have the wrong orientation. It can, however, be shown that it is not possible to have only one corner or edge piece oriented wrongly, so the presented move sequences will be enough to solve the orientations.

To achieve the last mentioned, we need to show that there is a certain *total twist* of the pieces that will always be preserved in any legal move. This total twist describes the sum of deviations of each corner or edge piece from the initial orientation, when they are in their correct positions. However, when the pieces are not in their correct positions, there is no natural way to define the "correct orientation". (This corresponds to the fact that the subgroup itself is the only coset that has a distinguishable "identity element".) By careful notation, we may decide beforehand what it means for each piece to be in "correct orientation" in any possible position. Then it is easy to show that the sum of deviations from this correct orientation is preserved under every basic move. Since the total twist is zero in the initial state, we can never have only one piece differing in orientation from the initial state.

8. Related concepts

During the course, it is also possible to talk about other subjects not perhaps directly related to the cube. For example, when introducing the concept of sign of a permutation, it is natural to spend a little time talking about the alternating groups, and maybe even go as far as mentioning their simplicity. Below are a few other examples, some of which have found their way to the lecture material.

Examples of symmetry groups. Permutations and the idea of "legal moves" can be used to describe all kinds of symmetry groups. As an example, it is convenient to present the dihedral groups and to talk about their properties: which "legal moves" are odd and which are even, what kind of conjugacy classes one finds, and so on.

Simplicity and solubility. The Rubik's group is divided into the orientational subgroup and the corresponding quotient group, namely the positional group. This division process could be taken further. (Actually this is exactly what is done implicitly with the positional group when it is divided into a direct product and its involutionary complement.) Finally we would end up with a composition series with simple factors. Some of these factors are alternating groups, and the others are cyclic. Groups with cyclic composition factors are soluble, and one can get an intuitive notion of solubility by considering the commutative parts of the puzzle "easier to solve".

Centralisers and centres. While talking about conjugation, it is natural to also mention those moves that stay put while conjugated by others. In the lectures, the theory of centralisers has been taken as far the class equation. On the cube, it is quite interesting to note (without proof) that the only non-identity state in the centre of the whole group requires as much as 20 right angle turns or half-turns to solve. This was recently proved to be the worst possible situation.

Commutator subgroup and abelianisation. Commutator subgroups and the corresponding quotients can be introduced at least in examples and exercises. Using knowledge obtained on the course, the students are for example able to find the commutator subgroups of the symmetric groups by themselves.

It is also interesting to look at other puzzles comparable to the cube. There are a lot a variations of the cube, for instance the $4\times4\times4$ and $5\times5\times5$

cubes, called "Rubik's Revenge" and "Professor's Cube", respectively. The group structures of these differ a little from the normal cube, but they offer good exercise material for the students. There are also more complicated structures, like the dodecahedral "Megaminx", whose group-theoretical solution is currently a Bachelor's (or Master's) thesis in progress. Even the problem of the normal cube itself can be augmented by marking the orientations of the centre pieces in the initial state and trying to restore those as well. This is known as the *supercube* problem.

There are also lots of other puzzles worth mentioning. Maybe the foremost of these is the "15 puzzle", which is also widely known. In this puzzle from the 1870's, a square is divided into 15 smaller squares together with one empty slot. Squares can be brought to the empty slot by sliding them past each other, and so the empty slot seems to be travelling across the large square. The aim is to bring the numbered squares into their original order. This puzzle is famous for its history, as for a time a reward was offered for solving it from a state that corresponds to an odd permutation, while only the even states are possible to solve.

The topic of different puzzles and their mathematical properties is very interesting and probably inexhaustible. In this way it resembles the Magic Cube with its multitude of possible states and solutions. At the realisation of this multitude, Ernő Rubik himself was astonished: before constructing his first model he had thought that the initial state could be easily restored. He could not have been more in the wrong.

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