Algebra II
Department of Mathematics and Statistics
Problem sheet 5
Thu 25.2.2010

1. Prove the following facts related to normal subgroups.

   (a) If $X, Y \leq G$ and $H \trianglelefteq X$, then $H \cap Y \trianglelefteq X \cap Y$.
   (b) If $H \trianglelefteq G$ and $K \trianglelefteq G$, then $HK \trianglelefteq G$.
   (c) If $X \trianglelefteq H$ and $f : G \to H$ is a homomorphism, then $f^{-1}X \trianglelefteq G$.
   (d) If $X \trianglelefteq G$ and $f : G \to H$ is a surjective homomorphism, then $fX \trianglelefteq H$.

2. Show that

   (a) every dihedral group is soluble
   (b) every finite p-group is soluble.

   Hint: In part (b), use the fact that a finite p-group has non-trivial centre.

3. Suppose that G is a group of order 80. Find its composition factors, and deduce that it is soluble.

4. Let G be a group. Show that the following conditions are equivalent:

   (i) G is simple and abelian.
   (ii) G is a finite cyclic group, whose order is a prime number.

5. Prove the *Fundamental Theorem of Arithmetic*: every integer $n > 1$ has a representation as a product of prime numbers, and this representation is unique up to a reordering of the factors.

   Hint: Use the Jordan-Hölder Theorem to the cyclic group C_n.

6. Suppose G is a group of order 10. Prove the following:

   (a) $H \trianglelefteq G$ for some $H \cong C_5$, and the composition factors of G are C_2 and C_5. (You can use Cauchy's Theorem.)
   (b) $K \leq G$ and $K \cap H = \{1\}$ for some $K \cong C_2$.
   (c) If $1 \neq a \in K$ and $b \in H$, then ${}^a b = b$ or ${}^a b = b^{-1}$.

   Conclude that there are only two different groups of order 10. What are they?